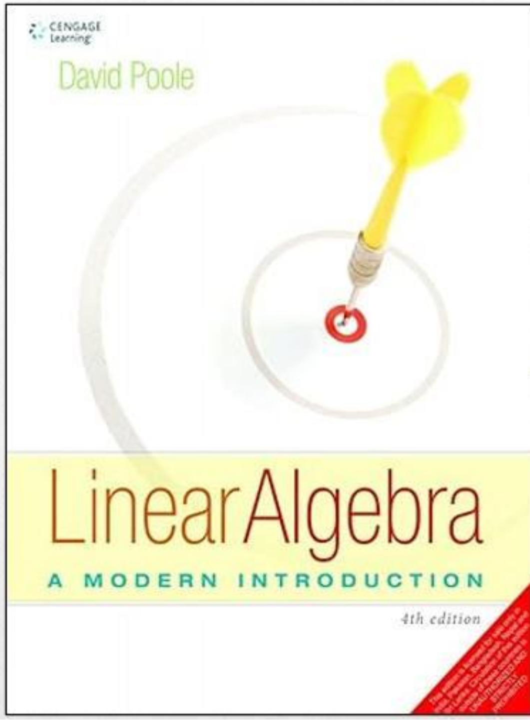

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線性代數 (二)

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7.1 Inner Product Spaces

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In Chapter 1, we defined the dot product $\mathbf{u} \cdot \mathbf{v}$ of vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$,

In this section, we will use the properties of the dot product as a means of defining the general notion of an *inner product.*

In the next section, we will show that inner products can be used to define analogues of "length" and "distance" in vector spaces other than $\mathbb{R}^n$.

Definition An ***inner product*** on a vector space V is an operation that assigns to every pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in V a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ such that the following properties hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and all scalars c:

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
2. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$
3. $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$
4. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$

A vector space with an inner product is called an ***inner product space.***

Example 7.1

$\mathbb{R}^n$ is an inner product space with $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$. Properties (1) through (4) were verified as Theorem 1.2.

Theorem 1.2

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$ and let c be a scalar. Then

a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ — Commutativity
b. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ — Distributivity
c. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$
d. $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

Definition An ***inner product*** on a vector space V is an operation that assigns to every pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in V a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ such that the following properties hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and all scalars c:

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
2. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$
3. $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$
4. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$

A vector space with an inner product is called an ***inner product space.***

Example 7.2

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be two vectors in $\mathbb{R}^2$. Show that

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2$$

defines an inner product.

Definition An ***inner product*** on a vector space V is an operation that assigns to every pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in V a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ such that the following properties hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and all scalars c:

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
2. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$
3. $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$
4. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$

A vector space with an inner product is called an ***inner product space.***

Solution We must verify properties (1) through (4).

Property (1) holds because

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2 = 2v_1u_1 + 3v_2u_2 = \langle \mathbf{v}, \mathbf{u} \rangle$$

If c is a scalar, then

$$\langle c\mathbf{u}, \mathbf{v} \rangle = 2(cu_1)v_1 + 3(cu_2)v_2$$
$$= c(2u_1v_1 + 3u_2v_2) = c\langle \mathbf{u}, \mathbf{v} \rangle$$

which verifies property (3).

Finally, $\langle \mathbf{u}, \mathbf{u} \rangle = 2u_1u_1 + 3u_2u_2 = 2u_1^2 + 3u_2^2 \geq 0$

and it is clear that $\langle \mathbf{u}, \mathbf{u} \rangle = 2u_1^2 + 3u_2^2 = 0$ if and only if

$u_1 = u_2 = 0$ This verifies property (4)

that $\langle \mathbf{u}, \mathbf{v} \rangle$, as defined, is an inner product.

Example 7.2 can be generalized to show that if $w_1, \ldots, w_n$ are *positive* scalars and

$$\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad \text{are vectors in } \mathbb{R}^n,$$

then

$$\langle \mathbf{u}, \mathbf{v} \rangle = w_1 u_1 v_1 + \cdots + w_n u_n v_n$$

defines an inner product on $\mathbb{R}^n$, called a ***weighted dot product.***

Recall that the dot product can be expressed as $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$.

Observe that we can write the weighted dot product in Equation (1) as

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T W \mathbf{v}$$

where W is the $n \times n$ diagonal matrix

$$W = \begin{bmatrix} w_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & w_n \end{bmatrix}$$

Example 7.3 Let A be a symmetric, positive definite $n \times n$ matrix (see Section 5.5) and let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Show that

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T A \mathbf{v}$$

defines an inner product.

Definition An ***inner product*** on a vector space V is an operation that assigns to every pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in V a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ such that the following properties hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and all scalars c:

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
2. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$
3. $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$
4. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$

A vector space with an inner product is called an ***inner product space.***

Solution We check that

$$\begin{aligned} \langle \mathbf{u}, \mathbf{v} \rangle &= \mathbf{u}^T A \mathbf{v} = \mathbf{u} \cdot A\mathbf{v} = A\mathbf{v} \cdot \mathbf{u} \\ &= A^T \mathbf{v} \cdot \mathbf{u} = (\mathbf{v}^T A)^T \cdot \mathbf{u} = \mathbf{v}^T A \mathbf{u} = \langle \mathbf{v}, \mathbf{u} \rangle \end{aligned}$$

To illustrate Example 7.3, let $A = \begin{bmatrix} 4 & -2 \\ -2 & 7 \end{bmatrix}$. Then

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T A \mathbf{v} = [u_1 \;\; u_2] \begin{bmatrix} 4 & -2 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 4u_1v_1 - 2u_1v_2 - 2u_2v_1 + 7u_2v_2$$

Finally, since A is positive definite, $\langle \mathbf{u}, \mathbf{u} \rangle = \mathbf{u}^T A \mathbf{u} > 0$ for all $\mathbf{u} \neq \mathbf{0}$,

so $\langle \mathbf{u}, \mathbf{u} \rangle = \mathbf{u}^T A \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Example 7.4 In $\mathscr{P}_2$, let $p(x) = a_0 + a_1x + a_2x^2$ and $q(x) = b_0 + b_1x + b_2x^2$. Show that

$$\langle p(x), q(x) \rangle = a_0b_0 + a_1b_1 + a_2b_2$$

defines an inner product on $\mathscr{P}_2$. (For example, if $p(x) = 1 - 5x + 3x^2$ and $q(x) = 6 + 2x - x^2$, then $\langle p(x), q(x) \rangle = 1 \cdot 6 + (-5) \cdot 2 + 3 \cdot (-1) = -7$.)

Solution Since $\mathscr{P}_2$ is isomorphic to $\mathbb{R}^3$, we need only show that the dot product in $\mathbb{R}^3$ is an inner product, which we have already established.

Example 7.4 In $\mathscr{P}_2$, let $p(x) = a_0 + a_1x + a_2x^2$ and $q(x) = b_0 + b_1x + b_2x^2$. Show that

$$\langle p(x), q(x)\rangle = a_0b_0 + a_1b_1 + a_2b_2$$

defines an inner product on $\mathscr{P}_2$. (For example, if $p(x) = 1 - 5x + 3x^2$ and $q(x) = 6 + 2x - x^2$, then $\langle p(x), q(x)\rangle = 1 \cdot 6 + (-5) \cdot 2 + 3 \cdot (-1) = -7$.)

Solution Since $\mathscr{P}_2$ is isomorphic to $\mathbb{R}^3$, we need only show that the dot product in $\mathbb{R}^3$ is an inner product, which we have already established.

Conclusion:

Since the operation $\langle p(x), q(x)\rangle = a_0b_0 + a_1b_1 + a_2b_2$ satisfies all four axioms (Symmetry, Additivity, Homogeneity, and Positivity), it is a valid inner product on $\mathscr{P}_2$.

Example 7.5 Let f and g be in $\mathscr{C}[a, b]$, the vector space of all continuous functions on the closed interval $[a, b]$. Show that

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx$$

defines an inner product on $\mathscr{C}[a, b]$.

$$\langle x^2, 1 + x\rangle = \int_0^1 x^2(1 + x)\,dx = \int_0^1 (x^2 + x^3)\,dx$$
$$= \left[\frac{x^3}{3} + \frac{x^4}{4}\right]_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

Solution We have

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx = \int_a^b g(x)f(x)\,dx = \langle g, f\rangle$$

If c is a scalar, then

$$\langle cf, g\rangle = \int_a^b cf(x)g(x)\,dx = c\int_a^b f(x)g(x)\,dx$$
$$= c\langle f, g\rangle$$

Finally, $\langle f, f\rangle = \int_a^b (f(x))^2\,dx \geq 0$, and it follows from

a theorem of calculus that,

since f is continuous, $\langle f, f\rangle = \int_a^b (f(x))^2\,dx = 0$

if and only if f is the zero function.

Therefore, $\langle f, g\rangle$ is an inner product on $\mathscr{C}[a, b]$.

Properties of Inner Products

Theorem 7.1 Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in an inner product space V and let c be a scalar.

a. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
b. $\langle \mathbf{u}, c\mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$
c. $\langle \mathbf{u}, \mathbf{0} \rangle = \langle \mathbf{0}, \mathbf{v} \rangle = 0$

Length, Distance, and Orthogonality

Definition Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V.

1. The ***length*** (or ***norm***) of $\mathbf{v}$ is $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.
2. The ***distance*** between $\mathbf{u}$ and $\mathbf{v}$ is $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$.
3. $\mathbf{u}$ and $\mathbf{v}$ are ***orthogonal*** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Note that $\|\mathbf{v}\|$ is always defined, since $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ by the definition of inner product, so we can take the square root of this nonnegative quantity.

As in $\mathbb{R}^n$, a vector of length 1
is called a ***unit vector.*** The ***unit sphere*** in V is the set S of all unit vectors in V.

Example 7.6 Consider the inner product on $\mathscr{C}[0, 1]$ given in Example 7.5. If $f(x) = x$ and $g(x) = 3x - 2$, find

(a) $\|f\|$ (b) $d(f, g)$ (c) $\langle f, g\rangle$

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx$$

Solution (a) We find that

$$\langle f, f\rangle = \int_0^1 f^2(x)\,dx = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big]_0^1 = \frac{1}{3}$$

so $\|f\| = \sqrt{\langle f, f\rangle} = 1/\sqrt{3}$.

(c) We compute

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx = \int_0^1 x(3x - 2)\,dx$$

$$= \int_0^1 (3x^2 - 2x)\,dx = [x^3 - x^2]_0^1 = 0$$

Thus, f and g are orthogonal.

Example 7.7 Using the inner product on $\mathbb{R}^2$ defined in Example 7.2, draw a sketch of the unit sphere (circle).

Solution If $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, then $\langle \mathbf{x}, \mathbf{x} \rangle = 2x^2 + 3y^2$. Since the unit sphere (circle) consists of all $\mathbf{x}$ such that $\|\mathbf{x}\| = 1$, we have

$$1 = \|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{2x^2 + 3y^2} \quad \text{or} \quad 2x^2 + 3y^2 = 1$$

This is the equation of an ellipse, and its graph is shown in Figure 7.2.

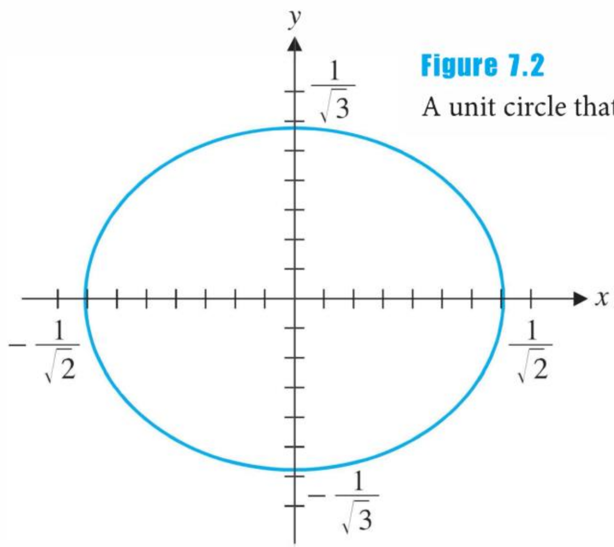

Figure 7.2
A unit circle that is an ellipse

Theorem 7.2 **Pythagoras' Theorem**

Let **u** and **v** be vectors in an inner product space V. Then **u** and **v** are orthogonal if and only if

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Proof As you will be asked to prove in Exercise 32, we have

$$\|\mathbf{u} + \mathbf{v}\|^2 = \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle = \|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2$$

It follows immediately that $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ if and only if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Orthogonal Projections and the Gram-Schmidt Process

an ***orthogonal set*** of vectors
in an inner product space V is a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of vectors from V such that
$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ whenever $\mathbf{v}_i \neq \mathbf{v}_j$.

An ***orthonormal set*** of vectors is then an orthogonal set of *unit* vectors.

An ***orthogonal basis*** for a subspace W of V is just a basis for W that is an orthogonal set;

an ***orthonormal basis*** for a subspace W of V is a basis for W that is an orthonormal set.

Example 7.8 Construct an orthogonal basis for $\mathscr{P}_2$ with respect to the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$$

by applying the Gram-Schmidt Process to the basis $\{1, x, x^2\}$.

Solution Let $\mathbf{x}_1 = 1$, $\mathbf{x}_2 = x$, and $\mathbf{x}_3 = x^2$. We begin by setting $\mathbf{v}_1 = \mathbf{x}_1 = 1$.
compute

$$\langle \mathbf{v}_1, \mathbf{v}_1\rangle = \int_{-1}^{1} dx = x\Big]_{-1}^{1} = 2 \quad \text{and} \quad \langle \mathbf{v}_1, \mathbf{x}_2\rangle = \int_{-1}^{1} x\,dx = \frac{x^2}{2}\Big]_{-1}^{1} = 0$$

Therefore, $\mathbf{v}_2 = \mathbf{x}_2 - \dfrac{\langle \mathbf{v}_1, \mathbf{x}_2\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\mathbf{v}_1 = x - \dfrac{0}{2}(1) = x$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{v}_1, \mathbf{x}_3\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\mathbf{v}_1 - \frac{\langle \mathbf{v}_2, \mathbf{x}_3\rangle}{\langle \mathbf{v}_2, \mathbf{v}_2\rangle}\mathbf{v}_2 = x^2 - \frac{\frac{2}{3}}{2}(1) - \frac{0}{\frac{2}{3}}x = x^2 - \frac{1}{3}$$

It follows that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for $\mathscr{P}_2$ on the interval $[-1, 1]$.

The polynomials 1, x, $x^2 - \frac{1}{3}$ are the first three ***Legendre polynomials.***

define the ***orthogonal projection*** $\text{proj}_W(\mathbf{v})$ of a vector $\mathbf{v}$ onto a subspace W of an inner product space. If $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis for W, then

$$\text{proj}_W(\mathbf{v}) = \frac{\langle \mathbf{u}_1, \mathbf{v} \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle}\mathbf{u}_1 + \cdots + \frac{\langle \mathbf{u}_k, \mathbf{v} \rangle}{\langle \mathbf{u}_k, \mathbf{u}_k \rangle}\mathbf{u}_k$$

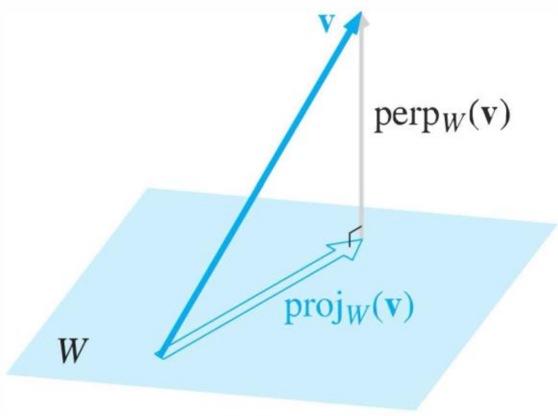

Figure 7.3

Then the ***component of* v *orthogonal to W*** is the vector

$$\text{perp}_W(\mathbf{v}) = \mathbf{v} - \text{proj}_W(\mathbf{v})$$

As in the Orthogonal Decomposition Theorem (Theorem 5.11), $\text{proj}_W(\mathbf{v})$ and $\text{perp}_W(\mathbf{v})$ are orthogonal (see Exercise 43),

The Cauchy-Schwarz and Triangle Inequalities

Theorem 7.3 **The Cauchy-Schwarz Inequality**

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V. Then

$$|\langle \mathbf{u}, \mathbf{v}\rangle| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$$

with equality holding if and only if $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other.

Proof If $\mathbf{u} = \mathbf{0}$, then the inequality is actually an equality, since $|\langle \mathbf{0}, \mathbf{v}\rangle| = 0 = \|\mathbf{0}\|\,\|\mathbf{v}\|$

If $\mathbf{u} \neq \mathbf{0}$, then let W be the subspace of V spanned by $\mathbf{u}$.

Since $\text{proj}_W(\mathbf{v}) = \dfrac{\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle}\mathbf{u}$ and $\text{perp}_W\,\mathbf{v} = \mathbf{v} - \text{proj}_W(\mathbf{v})$ are orthogonal,

we can apply Pythagoras' Theorem to obtain

$$\begin{aligned}\|\mathbf{v}\|^2 &= \|\text{proj}_W(\mathbf{v}) + (\mathbf{v} - \text{proj}_W(\mathbf{v}))\|^2 = \|\text{proj}_W(\mathbf{v}) + \text{perp}_W(\mathbf{v})\|^2 \\ &= \|\text{proj}_W(\mathbf{v})\|^2 + \|\text{perp}_W(\mathbf{v})\|^2\end{aligned}$$

It follows that $\|\text{proj}_W(\mathbf{v})\|^2 \le \|\mathbf{v}\|^2$.

Now $\|\text{proj}_W(\mathbf{v})\|^2 = \left\langle \dfrac{\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle}\mathbf{u}, \dfrac{\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle}\mathbf{u}\right\rangle = \left(\dfrac{\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle}\right)^2 \langle \mathbf{u}, \mathbf{u}\rangle = \dfrac{\langle \mathbf{u}, \mathbf{v}\rangle^2}{\langle \mathbf{u}, \mathbf{u}\rangle} = \dfrac{\langle \mathbf{u}, \mathbf{v}\rangle^2}{\|\mathbf{u}\|^2}$

so we have $\dfrac{\langle \mathbf{u}, \mathbf{v}\rangle^2}{\|\mathbf{u}\|^2} \le \|\mathbf{v}\|^2$ or, equivalently, $\langle \mathbf{u}, \mathbf{v}\rangle^2 \le \|\mathbf{u}\|^2\|\mathbf{v}\|^2$

Taking square roots, we obtain $|\langle \mathbf{u}, \mathbf{v}\rangle| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$.

Clearly this last inequality is an equality if and only if $\|\text{proj}_W(\mathbf{v})\|^2 = \|\mathbf{v}\|^2$.

$$\mathbf{v} = \text{proj}_W(\mathbf{v}) = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle}\mathbf{u}$$

If this is so, then $\mathbf{v}$ is a scalar multiple of $\mathbf{u}$. Conversely, if $\mathbf{v} = c\mathbf{u}$, then

$$\text{perp}_W(\mathbf{v}) = \mathbf{v} - \text{proj}_W(\mathbf{v}) = c\mathbf{u} - \frac{\langle \mathbf{u}, c\mathbf{u}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle}\mathbf{u} = c\mathbf{u} - \frac{c\langle \mathbf{u}, \mathbf{u}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle}\mathbf{u} = \mathbf{0}$$

so equality holds in the Cauchy-Schwarz Inequality.

Theorem 7.4 **The Triangle Inequality**

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V. Then

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

Proof Starting with the equality you will be asked to prove in Exercise 32, we have

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|^2 &= \|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2 \\ &\leq \|\mathbf{u}\|^2 + 2|\langle \mathbf{u}, \mathbf{v}\rangle| + \|\mathbf{v}\|^2 \\ &\leq \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2 && \text{by Cauchy-Schwarz} \\ &= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2 \end{aligned}$$

Taking square roots yields the result.

Exercises 7.1

2, 5, 7, 14, 18, 28, 32, 33