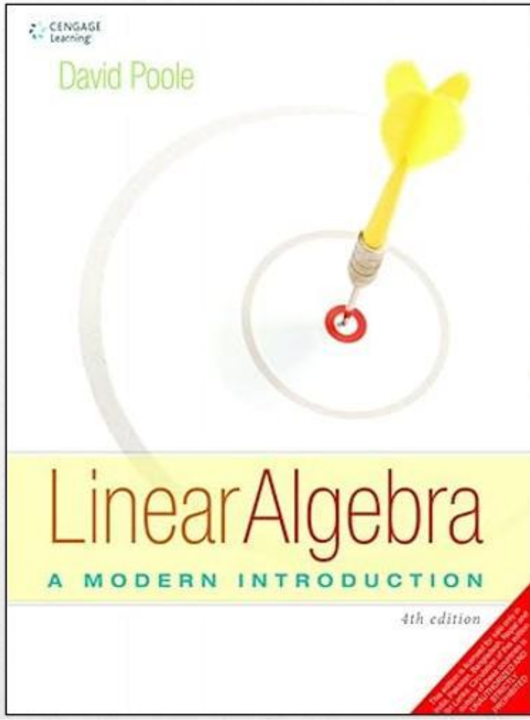

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線性代數 (二)

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6.6 The Matrix of a Linear Transformation

https://hmwu.idv.tw

Theorem 6.15 showed that a linear transformation $T : V \to W$ is completely determined by its effect on a spanning set for V.

Theorem 6.15 Let $T : V \to W$ be a linear transformation and let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a spanning set for V. Then $T(\mathcal{B}) = \{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ spans the range of T.

In particular, if we know how T acts on a basis for V, then we can compute $T(\mathbf{v})$ for any vector $\mathbf{v}$ in V.

We implicitly used this important property of linear transformations in Theorem 3.31 to help us compute the standard matrix of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$.

Theorem 3.31 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is a matrix transformation. More specifically, $T = T_A$, where A is the $m \times n$ matrix

$$A = [\,T(\mathbf{e}_1) \vdots T(\mathbf{e}_2) \vdots \cdots \vdots T(\mathbf{e}_n)\,]$$

In this section, we will show that every linear transformation between finite-dimensional vector spaces can be represented as a matrix transformation.

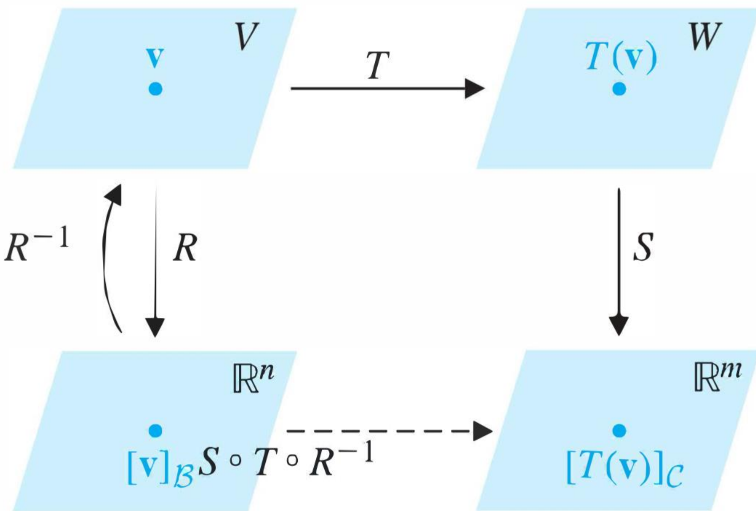

Figure 6.11

V is an n-dimensional vector space,

W is an m-dimensional vector space,

$T : V \to W$ is a linear transformation.

Let $\mathcal{B}$ and $\mathcal{C}$ be bases for V and W,

Then the coordinate vector mapping $R(\mathbf{v}) = [\mathbf{v}]_{\mathcal{B}}$ defines an isomorphism $R : V \to \mathbb{R}^n$.

we have an isomorphism $S : W \to \mathbb{R}^m$ given by $S(\mathbf{w}) = [\mathbf{w}]_{\mathcal{C}}$,

allows us to associate the image $T(\mathbf{v})$ with the vector $[T(\mathbf{v})]_{\mathcal{C}}$ in $\mathbb{R}^m$.

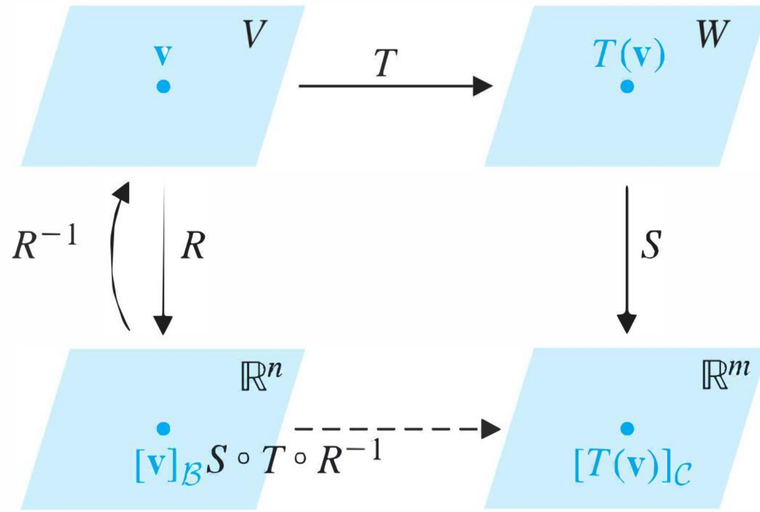

Figure 6.11

Since R is an isomorphism, it is invertible, so we may form the composite mapping

$$S \circ T \circ R^{-1} : \mathbb{R}^n \to \mathbb{R}^m \quad \text{which maps } [\mathbf{v}]_{\mathcal{B}} \text{ to } [T(\mathbf{v})]_{\mathcal{C}}.$$

Since this mapping goes from $\mathbb{R}^n$ to $\mathbb{R}^m$, that it is a matrix transformation.

What, then, is the standard matrix of $S \circ T \circ R^{-1}$?

We would like to find the $m \times n$ matrix A such that $A[\mathbf{v}]_{\mathcal{B}} = (S \circ T \circ R^{-1})([\mathbf{v}]_{\mathcal{B}})$.

Or, since $(S \circ T \circ R^{-1})([\mathbf{v}]_{\mathcal{B}}) = [T(\mathbf{v})]_{\mathcal{C}}$, we require $A[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{C}}$

It turns out to be surprisingly easy to find. The basic idea is that of Theorem 3.31.

Theorem 3.31 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is a matrix transformation. More specifically, $T = T_A$, where A is the $m \times n$ matrix

$$A = [T(\mathbf{e}_1) \vdots T(\mathbf{e}_2) \vdots \cdots \vdots T(\mathbf{e}_n)]$$

The columns of A are the images of the standard basis vectors for $\mathbb{R}^n$ under $S \circ T \circ R^{-1}$.

But, if $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for V, then

$$R(\mathbf{v}_i) = [\mathbf{v}_i]_{\mathcal{B}} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{e}_i \quad \leftarrow i\text{th entry}$$

so $R^{-1}(\mathbf{e}_i) = \mathbf{v}_i$.

Therefore, the ith column of the matrix A we seek is given by

$$(S \circ T \circ R^{-1})(\mathbf{e}_i) = S(T(R^{-1}(\mathbf{e}_i))) = S(T(\mathbf{v}_i)) = [T(\mathbf{v}_i)]_{\mathcal{C}}$$

which is the coordinate vector of $T(\mathbf{v}_i)$ with respect to the basis $\mathcal{C}$ of W.

Theorem 6.26 Let V and W be two finite-dimensional vector spaces with bases $\mathcal{B}$ and $\mathcal{C}$, respectively, where $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. If $T : V \to W$ is a linear transformation, then the $m \times n$ matrix A defined by

$$A = [\,[T(\mathbf{v}_1)]_{\mathcal{C}} \mid [T(\mathbf{v}_2)]_{\mathcal{C}} \mid \cdots \mid [T(\mathbf{v}_n)]_{\mathcal{C}}\,]$$

satisfies

$$A[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{C}}$$

for every vector $\mathbf{v}$ in V.

The matrix A is called the ***matrix of T with respect to the bases*** $\mathcal{B}$ ***and*** $\mathcal{C}$.

The relationship is illustrated below.

$$\begin{array}{ccc} \mathbf{v} & \xrightarrow{T} & T(\mathbf{v}) \\ \downarrow & & \downarrow \\ [\mathbf{v}]_{\mathcal{B}} & \xrightarrow{T_A} & A[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{C}} \end{array}$$

(Recall that T_A denotes multiplication by A.)

Remarks

- The matrix of a linear transformation T with respect to bases $\mathcal{B}$ and $\mathcal{C}$ is denoted by $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$.

  the final equation in Theorem 6.26 becomes $$[T]_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{C}}$$

  "The matrix for T times the coordinate vector for $\mathbf{v}$ gives the coordinate vector for $T(\mathbf{v})$."

In the special case where $V = W$ and $\mathcal{B} = \mathcal{C}$, we write $[T]_{\mathcal{B}}$ (instead of $[T]_{\mathcal{B}\leftarrow\mathcal{B}}$).

Theorem 6.26 then states that $$[T]_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{B}}$$

Remarks

- The matrix of a linear transformation with respect to given bases is unique.

  That is, for every vector $\mathbf{v}$ in V, there is only *one* matrix A with the property specified by Theorem 6.26—namely,
  $$A[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{C}}$$

Remarks

- The diagram that follows Theorem 6.26 is sometimes called a *commutative diagram* because we can start in the upper left-hand corner with the vector $\mathbf{v}$ and get to $[T(\mathbf{v})]_{\mathcal{C}}$ in the lower right-hand corner in two different, but equivalent, ways.

If we denote the coordinate mappings that map $\mathbf{v}$ to $[\mathbf{v}]_{\mathcal{B}}$ and $\mathbf{w}$ to $[\mathbf{w}]_{\mathcal{C}}$ by R and S,

then we can summarize this "commutativity" by $S \circ T = T_A \circ R$

The reason for the term *commutative* becomes clearer when $V = W$ and $\mathcal{B} = \mathcal{C}$, for

then $R = S$ too, and we have $R \circ T = T_A \circ R$

suggesting that the coordinate mapping R commutes with the linear transformation T (provided we use the matrix version of T—namely, $T_A = T_{[T]_{\mathcal{B}}}$—where it is required).

Remarks

- The matrix $[T]_{\mathcal{C} \leftarrow \mathcal{B}}$ depends on the *order* of the vectors in the bases $\mathcal{B}$ and $\mathcal{C}$. Rearranging the vectors within either basis will affect the matrix $[T]_{\mathcal{C} \leftarrow \mathcal{B}}$. [See Example 6.77(b).]

Example 6.76 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - 2y \\ x + y - 3z \end{bmatrix}$$

and let $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\mathcal{C} = \{\mathbf{e}_2, \mathbf{e}_1\}$ be bases for $\mathbb{R}^3$ and $\mathbb{R}^2$.

Find the matrix of T with respect to $\mathcal{B}$ and $\mathcal{C}$ and verify Theorem 6.26 for $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$.

Solution compute

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad T(\mathbf{e}_2) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad T(\mathbf{e}_3) = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

we need their coordinate vectors with respect to $\mathcal{C}$.

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{e}_2 + \mathbf{e}_1, \quad \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \mathbf{e}_2 - 2\mathbf{e}_1, \quad \begin{bmatrix} 0 \\ -3 \end{bmatrix} = -3\mathbf{e}_2 + 0\mathbf{e}_1$$

Therefore, the matrix of T with respect to $\mathcal{B}$ and $\mathcal{C}$ is

$$A = [T]_{\mathcal{C}\leftarrow\mathcal{B}} = [[T(\mathbf{e}_1)]_{\mathcal{C}} \ \ [T(\mathbf{e}_2)]_{\mathcal{C}} \ \ [T(\mathbf{e}_3)]_{\mathcal{C}}]$$

$$= \begin{bmatrix} 1 & 1 & -3 \\ 1 & -2 & 0 \end{bmatrix}$$

To verify Theorem 6.26 for $\mathbf{v}$, we first compute

$$T(\mathbf{v}) = T\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ 10 \end{bmatrix}$$

Using all of these facts, we confirm that

$$A[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & -3 \\ 1 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 10 \\ -5 \end{bmatrix} = [T(\mathbf{v})]_{\mathcal{C}}$$

Example 6.77

Let $D : \mathscr{P}_3 \to \mathscr{P}_2$ be the differential operator $D(p(x)) = p'(x)$. Let $\mathcal{B} = \{1, x, x^2, x^3\}$ and $\mathcal{C} = \{1, x, x^2\}$ be bases for $\mathscr{P}_3$ and $\mathscr{P}_2$, respectively.

(a) Find the matrix A of D with respect to $\mathcal{B}$ and $\mathcal{C}$.

Solution

(a) Since the images of the basis $\mathcal{B}$ under D are $D(1) = 0$, $D(x) = 1$, $D(x^2) = 2x$, and $D(x^3) = 3x^2$, their coordinate vectors with respect to $\mathcal{C}$ are

$$[D(1)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad [D(x)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [D(x^2)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \quad [D(x^3)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

Example 6.77

Let $D : \mathscr{P}_3 \to \mathscr{P}_2$ be the differential operator $D(p(x)) = p'(x)$. Let $\mathcal{B} = \{1, x, x^2, x^3\}$ and $\mathcal{C} = \{1, x, x^2\}$ be bases for $\mathscr{P}_3$ and $\mathscr{P}_2$, respectively.

(b) Find the matrix A' of D with respect to $\mathcal{B}'$ and $\mathcal{C}$, where $\mathcal{B}' = \{x^3, x^2, x, 1\}$.

Solution

(b) Since the basis $\mathcal{B}'$ is just $\mathcal{B}$ in the *reverse* order, we see that

$$A' = [D]_{\mathcal{C}\leftarrow\mathcal{B}'} = [\,[D(x^3)]_{\mathcal{C}} \mid [D(x^2)]_{\mathcal{C}} \mid [D(x)]_{\mathcal{C}} \mid [D(1)]_{\mathcal{C}}\,]$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix}$$

(This shows that the *order* of the vectors in the bases $\mathcal{B}$ and $\mathcal{C}$ affects the matrix of a transformation with respect to these bases.)

Example 6.77

Let $D : \mathcal{P}_3 \to \mathcal{P}_2$ be the differential operator $D(p(x)) = p'(x)$. Let $\mathcal{B} = \{1, x, x^2, x^3\}$ and $\mathcal{C} = \{1, x, x^2\}$ be bases for $\mathcal{P}_3$ and $\mathcal{P}_2$, respectively.

(c) Using part (a), compute $D(5 - x + 2x^3)$ and $D(a + bx + cx^2 + dx^3)$ to verify Theorem 6.26.

Solution

(c) First we compute $D(5 - x + 2x^3) = -1 + 6x^2$ directly,

getting the coordinate vector

$$[D(5 - x + 2x^3)]_{\mathcal{C}} = [-1 + 6x^2]_{\mathcal{C}} = \begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix}$$

so

$$A[5 - x + 2x^3]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix} = [D(5 - x + 2x^3)]_{\mathcal{C}}$$

which agrees with Theorem 6.26.

Example 6.78

Let $T: \mathscr{P}_2 \to \mathscr{P}_2$ be the linear transformation defined by

$$T(p(x)) = p(2x - 1)$$

(a) Find the matrix of T with respect to $\mathcal{E} = \{1, x, x^2\}$.

Solution (a) We see that

$$T(1) = 1, \quad T(x) = 2x - 1, \quad T(x^2) = (2x - 1)^2 = 1 - 4x + 4x^2$$

Therefore,

$$[T]_{\mathcal{E}} = [[T(1)]_{\mathcal{E}} \mid [T(x)]_{\mathcal{E}} \mid [T(x^2)]_{\mathcal{E}}] = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

Example 6.78

Let $T : \mathscr{P}_2 \to \mathscr{P}_2$ be the linear transformation defined by

$$T(p(x)) = p(2x - 1)$$

(b) Compute $T(3 + 2x - x^2)$ indirectly, using part (a).

(b) We apply Theorem 6.26 as follows:

The coordinate vector of $p(x) = 3 + 2x - x^2$ with respect to $\mathcal{E}$ is

$$[p(x)]_{\mathcal{E}} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

It follows that $T(3 + 2x - x^2) = 0 \cdot 1 + 8 \cdot x - 4 \cdot x^2 = 8x - 4x^2$.

[Verify this by computing $T(3 + 2x - x^2) = 3 + 2(2x - 1) - (2x - 1)^2$ directly.]

Example 6.79

Let $\mathscr{D}$ be the vector space of all differentiable functions. Consider the subspace W of $\mathscr{D}$ given by $W = \text{span}(e^{3x}, xe^{3x}, x^2e^{3x})$. Since the set $\mathcal{B} = \{e^{3x}, xe^{3x}, x^2e^{3x}\}$ is linearly independent (why?), it is a basis for W.

(a) Show that the differential operator D maps W into itself.

(b) Find the matrix of D with respect to $\mathcal{B}$.

Solution (a) Applying D to a general element of W, we see that

$$D(ae^{3x} + bxe^{3x} + cx^2e^{3x}) = (3a + b)e^{3x} + (3b + 2c)xe^{3x} + 3cx^2e^{3x}$$

(check this), which is again in W.

$$[D(e^{3x})]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \quad [D(xe^{3x})]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix},$$

$$[D(x^2e^{3x})]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$$

It follows that

$$[D]_{\mathcal{B}} = [[D(e^{3x})]_{\mathcal{B}} \mid [D(xe^{3x})]_{\mathcal{B}} \mid [D(x^2e^{3x})]_{\mathcal{B}}] = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Example 6.79

Let $\mathscr{D}$ be the vector space of all differentiable functions. Consider the subspace W of $\mathscr{D}$ given by $W = \text{span}(e^{3x}, xe^{3x}, x^2e^{3x})$. Since the set $\mathcal{B} = \{e^{3x}, xe^{3x}, x^2e^{3x}\}$ is linearly independent (why?), it is a basis for W.

(c) Compute the derivative of $5e^{3x} + 2xe^{3x} - x^2e^{3x}$ indirectly, using Theorem 6.26, and verify it using part (a).

(c) For $f(x) = 5e^{3x} + 2xe^{3x} - x^2e^{3x}$, we see by inspection that

$$[f(x)]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

which, in turn, implies that $f'(x) = D(f(x)) = 17e^{3x} + 4xe^{3x} - 3x^2e^{3x}$,

in agreement with the formula in part (a).

Example 6.80

Let V be an n-dimensional vector space and let I be the identity transformation on V. What is the matrix of I with respect to bases $\mathcal{B}$ and $\mathcal{C}$ of V if $\mathcal{B} = \mathcal{C}$ (including the order of the basis vectors)? What if $\mathcal{B} \neq \mathcal{C}$?

Solution Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Then $I(\mathbf{v}_1) = \mathbf{v}_1, \ldots, I(\mathbf{v}_n) = \mathbf{v}_n$, so

$$[I(\mathbf{v}_1)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{e}_1, \quad [I(\mathbf{v}_2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{e}_2, \quad \ldots, \quad [I(\mathbf{v}_n)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \mathbf{e}_n$$

In the case $\mathcal{B} \neq \mathcal{C}$, we have

$$[I(\mathbf{v}_1)]_{\mathcal{C}} = [\mathbf{v}_1]_{\mathcal{C}}, \quad \ldots, \quad [I(\mathbf{v}_n)]_{\mathcal{C}} = [\mathbf{v}_n]_{\mathcal{C}}$$

$$[I]_{\mathcal{C}\leftarrow\mathcal{B}} = [\,[\mathbf{v}_1]_{\mathcal{C}} \mid \cdots \mid [\mathbf{v}_n]_{\mathcal{C}}\,] = P_{\mathcal{C}\leftarrow\mathcal{B}}$$

the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$.

Matrices of Composite and Inverse Linear Transformations

Theorem 6.27 Let U, V, and W be finite-dimensional vector spaces with bases $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$, respectively. Let $T : U \to V$ and $S : V \to W$ be linear transformations. Then

$$[S \circ T]_{\mathcal{D} \leftarrow \mathcal{B}} = [S]_{\mathcal{D} \leftarrow \mathcal{C}}[T]_{\mathcal{C} \leftarrow \mathcal{B}}$$

Remarks

- In words, this theorem says, "The matrix of the composite is the product of the matrices."
- Notice how the "inner subscripts" $\mathcal{C}$ must match and appear to cancel each other out, leaving the "outer subscripts" in the form $\mathcal{D} \leftarrow \mathcal{B}$.

Example 6.81

Use matrix methods to compute $(S \circ T)\begin{bmatrix} a \\ b \end{bmatrix}$ for the linear transformations S and T of Example 6.56.

Solution Recall that $T : \mathbb{R}^2 \to \mathscr{P}_1$ and $S : \mathscr{P}_1 \to \mathscr{P}_2$ are defined by

$$T\begin{bmatrix} a \\ b \end{bmatrix} = a + (a + b)x \quad \text{and} \quad S(a + bx) = ax + bx^2$$

By Theorem 6.27, the matrix of $S \circ T$ with respect to $\mathcal{E}$ and $\mathcal{E}''$ is

$$\begin{aligned}[(S \circ T)]_{\mathcal{E}'' \leftarrow \mathcal{E}} &= [S]_{\mathcal{E}'' \leftarrow \mathcal{E}'}[T]_{\mathcal{E}' \leftarrow \mathcal{E}} \\ &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}\end{aligned}$$

Thus, by Theorem 6.26,

$$\begin{aligned}\left[(S \circ T)\begin{bmatrix} a \\ b \end{bmatrix}\right]_{\mathcal{E}''} &= [(S \circ T)]_{\mathcal{E}'' \leftarrow \mathcal{E}}\begin{bmatrix} a \\ b \end{bmatrix}_{\mathcal{E}} \\ &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ a + b \end{bmatrix}\end{aligned}$$

Consequently,

$$(S \circ T)\begin{bmatrix} a \\ b \end{bmatrix} = ax + (a + b)x^2$$

Theorem 6.28 Let $T : V \to W$ be a linear transformation between n-dimensional vector spaces V and W and let $\mathcal{B}$ and $\mathcal{C}$ be bases for V and W, respectively. Then T is invertible if and only if the matrix $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$ is invertible. In this case,

$$([T]_{\mathcal{C}\leftarrow\mathcal{B}})^{-1} = [T^{-1}]_{\mathcal{B}\leftarrow\mathcal{C}}$$

Example 6.82 In Example 6.70, the linear transformation $T : \mathbb{R}^2 \to \mathcal{P}_1$ defined by

$$T\begin{bmatrix} a \\ b \end{bmatrix} = a + (a + b)x$$

Solution In Example 6.81, $[T]_{\mathcal{E}'\leftarrow\mathcal{E}} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

By Theorem 6.28, it follows that the matrix of T^{-1} with respect to $\mathcal{E}'$ and $\mathcal{E}$ is

$$[T^{-1}]_{\mathcal{E}\leftarrow\mathcal{E}'} = ([T]_{\mathcal{E}'\leftarrow\mathcal{E}})^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

This means that

$$T^{-1}(a + bx) = a\mathbf{e}_1 + (b - a)\mathbf{e}_2 = \begin{bmatrix} a \\ b - a \end{bmatrix}$$

Example 6.83 Show that the differential operator, restricted to the subspace $W = \text{span}(e^{3x}, xe^{3x}, x^2e^{3x})$ of $\mathscr{D}$, is invertible, and use this fact to find the integral

$$\int x^2e^{3x}\,dx$$

Solution In Example 6.79, we found the matrix of D with respect to the basis $\mathcal{B} = \{e^{3x}, xe^{3x}, x^2e^{3x}\}$ of W to be

$$[D]_{\mathcal{B}} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Since integration is *antidifferentiation*, this is the matrix corresponding to integration on W. We want to integrate the function x^2e^{3x} whose coordinate vector is

$$[x^2e^{3x}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

by Theorem 6.26,

$$\begin{aligned} \left[\int x^2e^{3x}\,dx\right]_{\mathcal{B}} &= [D^{-1}(x^2e^{3x})]_{\mathcal{B}} \\ &= [D^{-1}]_{\mathcal{B}}[x^2e^{3x}]_{\mathcal{B}} \\ &= \begin{bmatrix} \frac{1}{3} & -\frac{1}{9} & \frac{2}{27} \\ 0 & \frac{1}{3} & -\frac{2}{9} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{27} \\ -\frac{2}{9} \\ \frac{1}{3} \end{bmatrix} \end{aligned}$$

It follows that

$$\int x^2e^{3x}\,dx = \tfrac{2}{27}e^{3x} - \tfrac{2}{9}xe^{3x} + \tfrac{1}{3}x^2e^{3x}$$

Change of Basis and Similarity

Theorem 6.29 Let V be a finite-dimensional vector space with bases $\mathcal{B}$ and $\mathcal{C}$ and let $T : V \to V$ be a linear transformation. Then

$$[T]_{\mathcal{C}} = P^{-1}[T]_{\mathcal{B}}P$$

where P is the change-of-basis matrix from $\mathcal{C}$ to $\mathcal{B}$.

$$[T]_{\mathcal{C}\leftarrow\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[T]_{\mathcal{B}\leftarrow\mathcal{B}}P_{\mathcal{B}\leftarrow\mathcal{C}}$$

Same Same

Same

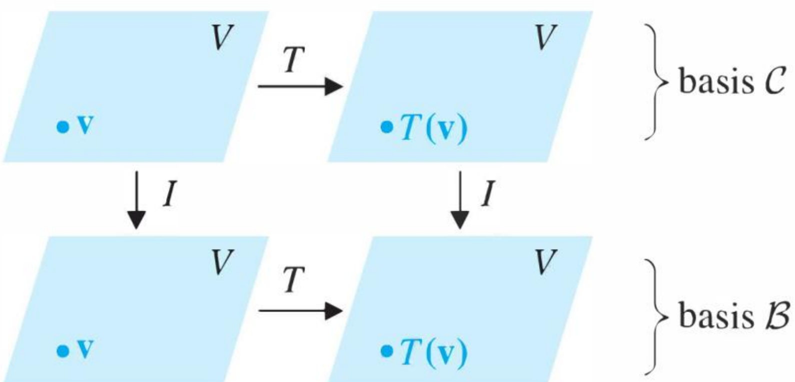

Figure 6.12
$I \circ T = T \circ I$

if the "upper" version of T is with respect to the basis $\mathcal{C}$ and the "lower" version is with respect to $\mathcal{B}$,

then $T = I \circ T = T \circ I$ is with respect to $\mathcal{C}$ in its domain and with respect to $\mathcal{B}$ in its codomain

With this notation, $P[T]_{\mathcal{C}\leftarrow\mathcal{C}} = [T]_{\mathcal{B}\leftarrow\mathcal{B}}P$

so $[T]_{\mathcal{C}\leftarrow\mathcal{C}} = P^{-1}[T]_{\mathcal{B}\leftarrow\mathcal{B}}P$ or $[T]_{\mathcal{C}} = P^{-1}[T]_{\mathcal{B}}P$

Thus, the matrices $[T]_{\mathcal{B}}$ and $[T]_{\mathcal{C}}$ are similar,

the matrix of T in this case is $[T]_{\mathcal{B}\leftarrow\mathcal{C}}$.

But $[T]_{\mathcal{B}\leftarrow\mathcal{C}} = [I \circ T]_{\mathcal{B}\leftarrow\mathcal{C}} = [I]_{\mathcal{B}\leftarrow\mathcal{C}}[T]_{\mathcal{C}\leftarrow\mathcal{C}}$

and $[T]_{\mathcal{B}\leftarrow\mathcal{C}} = [T \circ I]_{\mathcal{B}\leftarrow\mathcal{C}} = [T]_{\mathcal{B}\leftarrow\mathcal{B}}[I]_{\mathcal{B}\leftarrow\mathcal{C}}$

Therefore, $[I]_{\mathcal{B}\leftarrow\mathcal{C}}[T]_{\mathcal{C}\leftarrow\mathcal{C}} = [T]_{\mathcal{B}\leftarrow\mathcal{B}}[I]_{\mathcal{B}\leftarrow\mathcal{C}}$.

From Example 6.80, we know that $[I]_{\mathcal{B}\leftarrow\mathcal{C}} = P_{\mathcal{B}\leftarrow\mathcal{C}}$, the (invertible) change-of- basis matrix from $\mathcal{C}$ to $\mathcal{B}$.

If we denote this matrix by P, then we also have

$$P^{-1} = (P_{\mathcal{B}\leftarrow\mathcal{C}})^{-1} = P_{\mathcal{C}\leftarrow\mathcal{B}}$$

Example 6.84

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 3y \\ 2x + 2y \end{bmatrix}$$

Solution The matrix of T with respect to the standard basis $\mathcal{E}$ is

$$[T]_{\mathcal{E}} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$$ is diagonalizable Example 4.24.

If we let $\mathcal{C}$ be the basis of $\mathbb{R}^2$ consisting of the columns of P, then

P is the change-of-basis matrix $P_{\mathcal{E}\leftarrow\mathcal{C}}$ from $\mathcal{C}$ to $\mathcal{E}$.

By Theorem 6.29, $[T]_{\mathcal{C}} = P^{-1}[T]_{\mathcal{E}}P = D$

so the matrix of T with respect to the basis $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right\}$ is diagonal.

Remarks

It is easy to check that the solution above is correct by computing $[T]_{\mathcal{C}}$ directly.

$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$T\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

the coordinate vectors that form the columns of $[T]_{\mathcal{C}}$ are

$$\left[T\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]_{\mathcal{C}} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$ and

$$\left[T\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right]_{\mathcal{C}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

in agreement with our solution.

Definition Let V be a finite-dimensional vector space and let $T: V \to V$ be a linear transformation. Then T is called ***diagonalizable*** if there is a basis $\mathcal{C}$ for V such that the matrix $[T]_{\mathcal{C}}$ is a diagonal matrix.

Example 6.85

Let ℓ be the line through the origin in $\mathbb{R}^2$ with direction vector $\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$. Find the standard matrix of the projection onto ℓ.

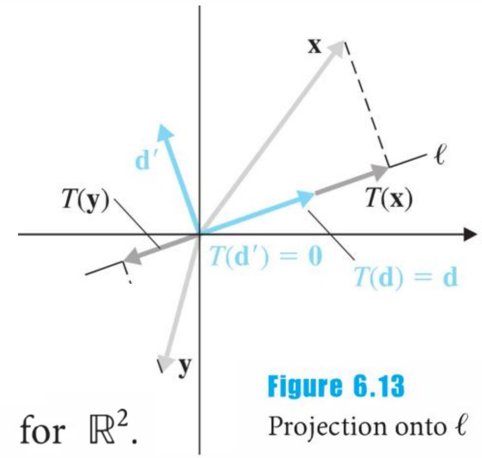

Figure 6.13
Projection onto ℓ

Solution Let T denote the projection.
assuming that $\mathbf{d}$ is a unit vector (i.e., $d_1^2 + d_2^2 = 1$),

since any nonzero multiple of $\mathbf{d}$ can serve as a direction vector for ℓ.

Let $\mathbf{d}' = \begin{bmatrix} -d_2 \\ d_1 \end{bmatrix}$ so that $\mathbf{d}$ and $\mathbf{d}'$ are orthogonal.
Since $\mathbf{d}'$ is also a unit vector, the set $\mathcal{D} = \{\mathbf{d}, \mathbf{d}'\}$ is an orthonormal basis for $\mathbb{R}^2$.

As Figure 6.13 shows, $T(\mathbf{d}) = \mathbf{d}$ and $T(\mathbf{d}') = \mathbf{0}$. Therefore,

$$[T(\mathbf{d})]_{\mathcal{D}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad [T(\mathbf{d}')]_{\mathcal{D}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{so} \quad [T]_{\mathcal{D}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

By Theorem 6.29,

$$\begin{aligned}[T]_{\mathcal{E}} &= P_{\mathcal{E}\leftarrow\mathcal{D}}[T]_{\mathcal{D}}P_{\mathcal{D}\leftarrow\mathcal{E}} \\ &= \begin{bmatrix} d_1 & -d_2 \\ d_2 & d_1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} d_1 & d_2 \\ -d_2 & d_1 \end{bmatrix} = \begin{bmatrix} d_1^2 & d_1 d_2 \\ d_1 d_2 & d_2^2 \end{bmatrix}\end{aligned}$$

which agrees with part (b) of Example 3.59.

Example 6.86

Let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be the linear transformation defined by

$$T(p(x)) = p(2x - 1)$$

(a) Find the matrix of T with respect to the basis $\mathcal{B} = \{1 + x, 1 - x, x^2\}$ of $\mathcal{P}_2$.

Solution (a) In Example 6.78, we found that the matrix of T with respect to the standard basis $\mathcal{E} = \{1, x, x^2\}$ is

$$[T]_{\mathcal{E}} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

It follows that the matrix of T with respect to $\mathcal{B}$ is $[T]_{\mathcal{B}} = P^{-1}[T]_{\mathcal{E}}P$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ -1 & 2 & \frac{5}{2} \\ 0 & 0 & 4 \end{bmatrix}$$

Example 6.86

Let $T : \mathscr{P}_2 \to \mathscr{P}_2$ be the linear transformation defined by

$$T(p(x)) = p(2x - 1)$$

(b) Show that T is diagonalizable and find a basis $\mathcal{C}$ for $\mathscr{P}_2$ such that $[T]_{\mathcal{C}}$ is a diagonal matrix.

(b) The eigenvalues of $[T]_{\mathcal{E}}$ are 1, 2, and 4 (why?), so we know from Theorem 4.25 that $[T]_{\mathcal{E}}$ is diagonalizable.

Eigenvectors corresponding to these eigenvalues are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Furthermore, P is the change-of-basis matrix from a basis $\mathcal{C}$ to $\mathcal{E}$,

and the columns of P are thus the coordinate vectors of $\mathcal{C}$ in terms of $\mathcal{E}$.

It follows that $\mathcal{C} = \{1, -1 + x, 1 - 2x + x^2\}$ and $[T]_{\mathcal{C}} = D$.

Theorem 6.30 The Fundamental Theorem of Invertible Matrices: Version 4

Let A be an $n \times n$ matrix and let $T: V \to W$ be a linear transformation whose matrix $[T]_{\mathcal{C} \leftarrow \mathcal{B}}$ with respect to bases $\mathcal{B}$ and $\mathcal{C}$ of V and W, respectively, is A. The following statements are equivalent:

a. A is invertible.
b. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.
c. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
d. The reduced row echelon form of A is I_n.
e. A is a product of elementary matrices.
f. $\text{rank}(A) = n$
g. $\text{nullity}(A) = 0$
h. The column vectors of A are linearly independent.
i. The column vectors of A span $\mathbb{R}^n$.
j. The column vectors of A form a basis for $\mathbb{R}^n$.
k. The row vectors of A are linearly independent.
l. The row vectors of A span $\mathbb{R}^n$.
m. The row vectors of A form a basis for $\mathbb{R}^n$.
n. $\det A \neq 0$
o. 0 is not an eigenvalue of A.
p. T is invertible.
q. T is one-to-one.
r. T is onto.
s. $\ker(T) = \{\mathbf{0}\}$
t. $\text{range}(T) = W$

Exercises 6.6

3, 6, 9, 17, 21, 24, 32, 33