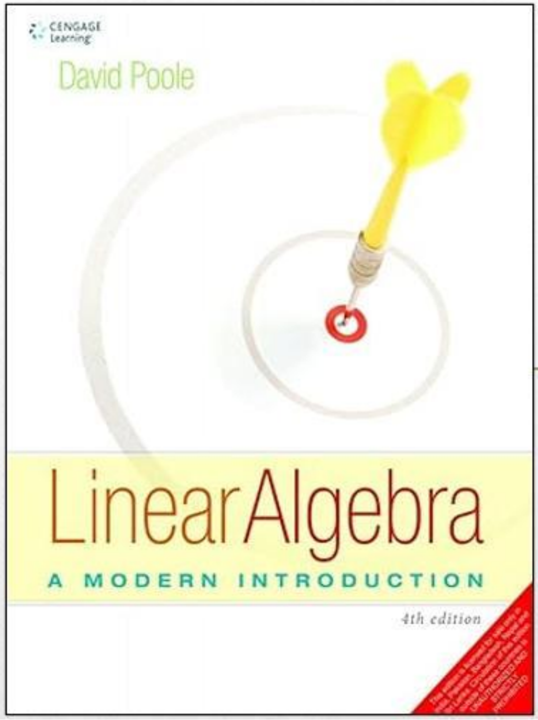
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線性代數 (二)

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6.5 The Kernel and Range of a Linear Transformation

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The null space and column space are two of the fundamental subspaces associated with a matrix. In this section, we extend these notions to the kernel and range of a linear transformation.

Definition Let $T: V \to W$ be a linear transformation. The ***kernel*** of T, denoted $\ker(T)$, is the set of all vectors in V that are mapped by T to $\mathbf{0}$ in W. That is,

$$\ker(T) = \{\mathbf{v} \text{ in } V: T(\mathbf{v}) = \mathbf{0}\}$$

The ***range*** of T, denoted $\text{range}(T)$, is the set of all vectors in W that are images of vectors in V under T. That is,

$$\begin{aligned}\text{range}(T) &= \{T(\mathbf{v}) : \mathbf{v} \text{ in } V\} \\ &= \{\mathbf{w} \text{ in } W : \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \text{ in } V\}\end{aligned}$$

Example 6.59 Let A be an $m \times n$ matrix and let $T = T_A$ be the corresponding matrix transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined by $T(\mathbf{v}) = A\mathbf{v}$.

the range of T is the column space of A.

The kernel of T is $\ker(T) = \{\mathbf{v} \text{ in } \mathbb{R}^n : T(\mathbf{v}) = \mathbf{0}\} = \{\mathbf{v} \text{ in } \mathbb{R}^n : A\mathbf{v} = \mathbf{0}\} = \text{null}(A)$

In words, the kernel of a matrix transformation is just the null space of the corresponding matrix.

Example 6.60 Find the kernel and range of the differential operator $D : \mathcal{P}_3 \to \mathcal{P}_2$ defined by $D(p(x)) = p'(x)$.

But $b + 2cx + 3dx^2 = 0$ if and only if $b = 2c = 3d = 0$, which implies that

$b = c = d = 0$. Therefore,

$$\ker(D) = \{a + bx + cx^2 + dx^3 : b = c = d = 0\} = \{a : a \text{ in } \mathbb{R}\}$$

In other words, the kernel of D is the set of constant polynomials.

The range of D is all of $\mathcal{P}_2$, since *every* polynomial in $\mathcal{P}_2$ is the image under D (i.e., the derivative) of *some* polynomial in $\mathcal{P}_3$.

To be specific, if $a + bx + cx^2$ is in $\mathcal{P}_2$, then $a + bx + cx^2 = D\left(ax + \left(\frac{b}{2}\right)x^2 + \left(\frac{c}{3}\right)x^3\right)$

Example 6.61

Let $S : \mathscr{P}_1 \to \mathbb{R}$ be the linear transformation defined by

$$S(p(x)) = \int_0^1 p(x)\,dx$$

Find the kernel and range of S.

Solution

$$S(a + bx) = \int_0^1 (a + bx)\,dx = \left[ax + \frac{b}{2}x^2\right]_0^1 = \left(a + \frac{b}{2}\right) - 0 = a + \frac{b}{2}$$

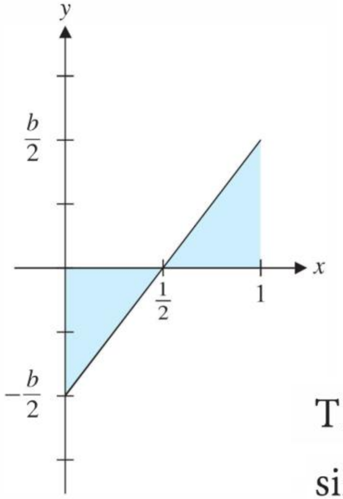


Figure 6.7

If $y = -\frac{b}{2} + bx$,

then $\int_0^1 y\,dx = 0$

Geometrically, ker(S) consists of all those linear polynomials whose graphs have the property that the area between the line and the x-axis is equally distributed above and below the axis on the interval [0, 1] (see Figure 6.7).

The range of S is $\mathbb{R}$,

since every real number can be obtained as the image under S of some polynomial in $\mathscr{P}_1$.

For example, if a is an arbitrary real number, then

$$\int_0^1 a\,dx = [ax]_0^1 = a - 0 = a$$

so $a = S(a)$.

Example 6.62 Let $T : M_{22} \to M_{22}$ be the linear transformation defined by taking transposes: $T(A) = A^T$. Find the kernel and range of T.

But if $A^T = O$, then $A = (A^T)^T = O^T = O$. It follows that $\ker(T) = \{O\}$.

Since, for any matrix A in M_{22},

we have $A = (A^T)^T = T(A^T)$ (and A^T is in M_{22}),

we deduce that $\operatorname{range}(T) = M_{22}$.

In all of these examples, the kernel and range of a linear transformation are subspaces of the domain and codomain, respectively, of the transformation.

Theorem 6.18 Let $T: V \to W$ be a linear transformation. Then:

a. The kernel of T is a subspace of V.
b. The range of T is a subspace of W.

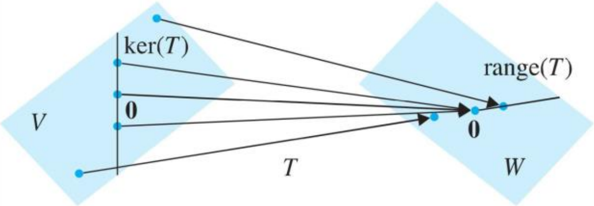
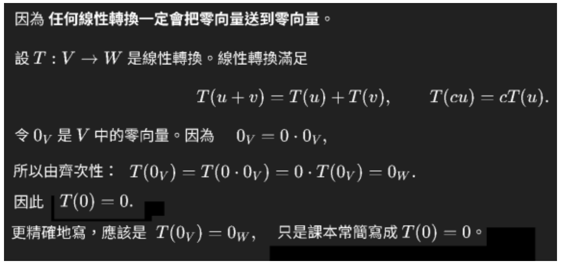


Figure 6.8
The kernel and range of $T: V \to W$



因為 **任何線性轉換一定會把零向量送到零向量**。

設 $T : V \to W$ 是線性轉換。線性轉換滿足

$$T(u+v) = T(u) + T(v), \qquad T(cu) = cT(u).$$

令 0_V 是 V 中的零向量。因為 $0_V = 0 \cdot 0_V$,

所以由齊次性： $T(0_V) = T(0 \cdot 0_V) = 0 \cdot T(0_V) = 0_W$.

因此 $T(0) = 0.$

更精確地寫，應該是 $T(0_V) = 0_W$, 只是課本常簡寫成 $T(0) = 0$。

(b) Since $\mathbf{0} = T(\mathbf{0})$, the zero vector of W is in range(T), so range(T) is nonempty.

Let $T(\mathbf{u})$ and $T(\mathbf{v})$ be in the range of T and let c be a scalar.
Then $T(\mathbf{u}) + T(\mathbf{v}) = T(\mathbf{u} + \mathbf{v})$ is the image of the vector $\mathbf{u} + \mathbf{v}$.

Since $\mathbf{u}$ and $\mathbf{v}$ are in V, so is $\mathbf{u} + \mathbf{v}$, and hence $T(\mathbf{u}) + T(\mathbf{v})$ is in range (T).

Similarly, $cT(\mathbf{u}) = T(c\mathbf{u})$. Since $\mathbf{u}$ is in V, so is $c\mathbf{u}$, and hence $cT(\mathbf{u})$ is in range(T).

Therefore, range(T) is a nonempty subset of W that is closed under addition and scalar multiplication, and thus it is a subspace of W.

In Chapter 3, we defined the rank of a matrix to be the dimension of its column space and the nullity of a matrix to be the dimension of its null space. We now extend these definitions to linear transformations.

Definition Let $T : V \to W$ be a linear transformation. The ***rank*** of T is the dimension of the range of T and is denoted by $\operatorname{rank}(T)$. The ***nullity*** of T is the dimension of the kernel of T and is denoted by $\operatorname{nullity}(T)$.

Example 6.63

If A is a matrix and $T = T_A$ is the matrix transformation defined by $T(\mathbf{v}) = A\mathbf{v}$, then the range and kernel of T are the column space and the null space of A, respectively, by Example 6.59. Hence, from Section 3.5, we have

$$\operatorname{rank}(T) = \operatorname{rank}(A) \quad \text{and} \quad \operatorname{nullity}(T) = \operatorname{nullity}(A)$$

Example 6.64

Find the rank and the nullity of the linear transformation $D : \mathscr{P}_3 \to \mathscr{P}_2$ defined by $D(p(x)) = p'(x)$.

Solution In Example 6.60, we computed range $(D) = \mathscr{P}_2$, so

$$\operatorname{rank}(D) = \dim \mathscr{P}_2 = 3$$

Example 6.65

Find the rank and the nullity of the linear transformation $S : \mathscr{P}_1 \to \mathbb{R}$ defined by

$$S(p(x)) = \int_0^1 p(x)\,dx$$

Solution From Example 6.61, range$(S) = \mathbb{R}$ and rank$(S) = \dim \mathbb{R} = 1$.

Also,

$$\begin{aligned}\ker(S) &= \left\{ -\frac{b}{2} + bx : b \text{ in } \mathbb{R} \right\} \\ &= \{ b(-\tfrac{1}{2} + x) : b \text{ in } \mathbb{R} \} \\ &= \operatorname{span}(-\tfrac{1}{2} + x)\end{aligned}$$

so $\{-\frac{1}{2} + x\}$ is a basis for ker(S). Therefore, nullity$(S) = \dim(\ker(S)) = 1$.

Example 6.66

Find the rank and the nullity of the linear transformation $T : M_{22} \to M_{22}$ defined by $T(A) = A^T$.

In Chapter 3, we saw that the rank and nullity of an $m \times n$ matrix A are related by the formula $\operatorname{rank}(A) + \operatorname{nullity}(A) = n$. This is the Rank Theorem (Theorem 3.26).

Since the matrix transformation $T = T_A$ has $\mathbb{R}^n$ as its domain, we could rewrite the relationship as $\operatorname{rank}(A) + \operatorname{nullity}(A) = \dim \mathbb{R}^n$

Theorem 6.19 **The Rank Theorem**

Let $T : V \to W$ be a linear transformation from a finite-dimensional vector space V into a vector space W. Then

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim V$$

Theorem 6.19 **The Rank Theorem**

Let $T : V \to W$ be a linear transformation from a finite-dimensional vector space V into a vector space W. Then

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim V$$

Proof

Let $\dim V = n$ and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a basis for $\ker(T)$

[so that $\operatorname{nullity}(T) = \dim(\ker(T)) = k$].

Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a linearly independent set, it can be extended to a basis for V, by Theorem 6.28.

Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ be such a basis.

If we can show that the set $\mathcal{C} = \{T(\mathbf{v}_{k+1}), \ldots, T(\mathbf{v}_n)\}$ is a basis for range(T), then we will have $\operatorname{rank}(T) = \dim(\operatorname{range}(T)) = n - k$ and thus

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = k + (n - k) = n = \dim V$$ as required.

Certainly $\mathcal{C}$ is contained in the range of T. To show that $\mathcal{C}$ spans the range of T, let $T(\mathbf{v})$ be a vector in the range of T.

Then $\mathbf{v}$ is in V, and since $\mathcal{B}$ is a basis for V, we can find scalars $c_1, \ldots, c_n$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k + c_{k+1}\mathbf{v}_{k+1} + \cdots + c_n\mathbf{v}_n$$

Since $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are in the kernel of T, we have $T(\mathbf{v}_1) = \cdots = T(\mathbf{v}_k) = \mathbf{0}$, so

$$\begin{aligned} T(\mathbf{v}) &= T(c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k + c_{k+1}\mathbf{v}_{k+1} + \cdots + c_n\mathbf{v}_n) \\ &= c_1T(\mathbf{v}_1) + \cdots + c_kT(\mathbf{v}_k) + c_{k+1}T(\mathbf{v}_{k+1}) + \cdots + c_nT(\mathbf{v}_n) \\ &= c_{k+1}T(\mathbf{v}_{k+1}) + \cdots + c_nT(\mathbf{v}_n) \end{aligned}$$

This shows that the range of T is spanned by $\mathcal{C}$.

Theorem 6.19 **The Rank Theorem**

Let $T : V \to W$ be a linear transformation from a finite-dimensional vector space V into a vector space W. Then

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim V$$

To show that $\mathcal{C}$ is linearly independent,

suppose that there are scalars $c_{k+1}, \ldots, c_n$ such that $c_{k+1}T(\mathbf{v}_{k+1}) + \cdots + c_nT(\mathbf{v}_n) = \mathbf{0}$

Then $T(c_{k+1}\mathbf{v}_{k+1} + \cdots + c_n\mathbf{v}_n) = \mathbf{0}$, which means that $c_{k+1}\mathbf{v}_{k+1} + \cdots + c_n\mathbf{v}_n$ is in the kernel of T and is, hence, expressible as a linear combination of the basis vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ of $\ker(T)$

$$c_{k+1}\mathbf{v}_{k+1} + \cdots + c_n\mathbf{v}_n = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$$

But now $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k - c_{k+1}\mathbf{v}_{k+1} - \cdots - c_n\mathbf{v}_n = \mathbf{0}$
and the linear independence of $\mathcal{B}$ forces $c_1 = \cdots = c_n = 0$.

In particular, $c_{k+1} = \cdots = c_n = 0$, which means $\mathcal{C}$ is linearly independent.

We have shown that $\mathcal{C}$ is a basis for the range of T, so,
by our comments above, the proof is complete.

Example 6.67

Find the rank and nullity of the linear transformation $T : \mathscr{P}_2 \to \mathscr{P}_3$ defined by $T(p(x)) = xp(x)$. (Check that T really is linear.)

Solution we have

$$T(a + bx + cx^2) = ax + bx^2 + cx^3$$

so we have nullity$(T) = \dim(\ker(T)) = 0$.

The Rank Theorem implies that $\operatorname{rank}(T) = \dim \mathscr{P}_2 - \operatorname{nullity}(T) = 3 - 0 = 3$

Example 6.68

Let W be the vector space of all symmetric 2×2 matrices. Define a linear transformation $T : W \to \mathscr{P}_2$ by

$$T\begin{bmatrix} a & b \\ b & c \end{bmatrix} = (a - b) + (b - c)x + (c - a)x^2$$

(Check that T is linear.) Find the rank and nullity of T.

$$= \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : (a - b) + (b - c)x + (c - a)x^2 = 0 \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : (a - b) = (b - c) = (c - a) = 0 \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : a = b = c \right\} = \left\{ \begin{bmatrix} c & c \\ c & c \end{bmatrix} \right\} = \operatorname{span}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right)$$

Therefore, $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ is a basis for the kernel of T, so $\operatorname{nullity}(T) = \dim(\ker(T)) = 1$.

The Rank Theorem tell us that $\operatorname{rank}(T) = \dim W - \operatorname{nullity}(T) = 3 - 1 = 2$.

One-to-One and Onto Linear Transformations

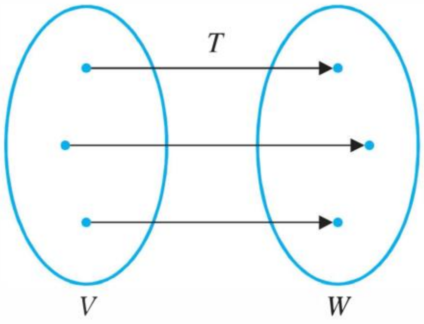
Definition A linear transformation $T: V \rightarrow W$ is called ***one-to-one*** if T maps distinct vectors in V to distinct vectors in W. If range$(T) = W$, then T is called ***onto.***

$T: V \rightarrow W$ is one-to-one if, for all $\mathbf{u}$ and $\mathbf{v}$ in V,

$$\mathbf{u} \neq \mathbf{v} \text{ implies that } T(\mathbf{u}) \neq T(\mathbf{v})$$

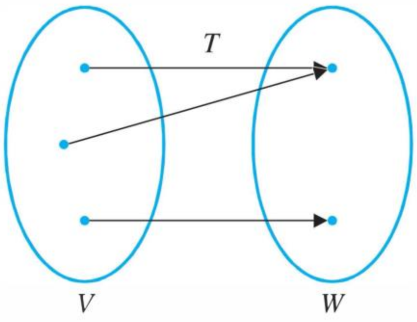
$T: V \rightarrow W$ is one-to-one if, for all $\mathbf{u}$ and $\mathbf{v}$ in V,

$$T(\mathbf{u}) = T(\mathbf{v}) \text{ implies that } \mathbf{u} = \mathbf{v}$$



(a) T is one-to-one

Figure 6.9

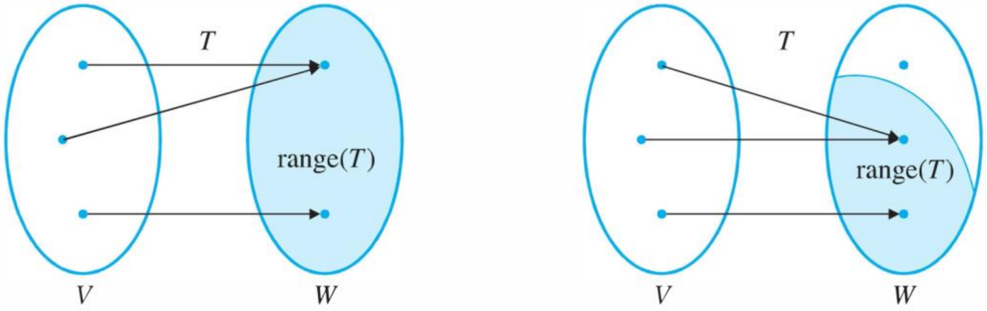


(b) T is not one-to-one

在線性代數中，**Onto (映成/滿射)** 指的是線性變換 T: $V \rightarrow W$ 的值域（Range/Image）等於對應域（Codomain）W。這意味著對應域 W 中的每一個向量都能被對應到。在矩陣形式 $Ax = b$ 中，若 A 為 onto，表示對於任何 b，線性方程組永遠都有解。

$T : V \to W$ is onto if, for all $\mathbf{w}$ in W, there is at least one $\mathbf{v}$ in V such that

$$\mathbf{w} = T(\mathbf{v})$$



(a) T is onto (b) T is not onto

Figure 6.10

Example 6.69

Which of the following linear transformations are one-to-one? onto?

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ x - y \\ 0 \end{bmatrix}$

T is not onto, since its range is not all of $\mathbb{R}^3$.

To be specific, there is no vector $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ such that $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. (Why not?)

Example 6.69

Which of the following linear transformations are one-to-one? onto?

(b) $D : \mathscr{P}_3 \to \mathscr{P}_2$ defined by $D(p(x)) = p'(x)$

(c) $T : M_{22} \to M_{22}$ defined by $T(A) = A^T$

(b) In Example 6.60, we showed that $\text{range}(D) = \mathscr{P}_2$, so D is onto.

D is not one-to-one, since distinct polynomials in $\mathscr{P}_3$ can have the same derivative.

For example, $x^3 \neq x^3 + 1$, but $D(x^3) = 3x^2 = D(x^3 + 1)$.

(c) Let A and B be in M_{22}, with $T(A) = T(B)$.

Then $A^T = B^T$, so $A = (A^T)^T = (B^T)^T = B$. Hence, T is one-to-one.

In Example 6.62, we showed that $\text{range}(T) = M_{22}$. Hence, T is onto.

Theorem 6.20 A linear transformation $T: V \to W$ is one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$.

Conversely, assume that $\ker(T) = \{\mathbf{0}\}$.

To show that T is one-to-one, let $\mathbf{u}$ and $\mathbf{v}$ be in V with $T(\mathbf{u}) = T(\mathbf{v})$.

Then $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v}) = \mathbf{0}$, which implies that $\mathbf{u} - \mathbf{v}$ is in the kernel of T.

But $\ker(T) = \{\mathbf{0}\}$, so we must have $\mathbf{u} - \mathbf{v} = \mathbf{0}$ or, equivalently, $\mathbf{u} = \mathbf{v}$.

This proves that T is one-to-one.

Example 6.70

Show that the linear transformation $T : \mathbb{R}^2 \to \mathscr{P}_1$ defined by

$$T\begin{bmatrix} a \\ b \end{bmatrix} = a + (a + b)x$$

is one-to-one and onto.

By the Rank Theorem, $\operatorname{rank}(T) = \dim \mathbb{R}^2 - \operatorname{nullity}(T) = 2 - 0 = 2$

Therefore, the range of T is a two-dimensional subspace of $\mathbb{R}^2$, and hence $\operatorname{range}(T) = \mathbb{R}^2$. It follows that T is onto.

Theorem 6.21 Let $\dim V = \dim W = n$. Then a linear transformation $T : V \to W$ is one-to-one if and only if it is onto.

Proof Assume that T is one-to-one. Then $\text{nullity}(T) = 0$ by Theorem 6.20

The Rank Theorem implies that

$$\text{rank}(T) = \dim V - \text{nullity}(T) = n - 0 = n$$

Therefore, T is onto.

Conversely, assume that T is onto. Then $\text{rank}(T) = \dim W = n$.

By the Rank Theorem,

$$\text{nullity}(T) = \dim V - \text{rank}(T) = n - n = 0$$

Hence, $\ker(T) = \{\mathbf{0}\}$, and T is one-to-one.

Theorem 6.22 Let $T : V \to W$ be a one-to-one linear transformation. If $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a linearly independent set in V, then $T(S) = \{T(\mathbf{v}_1), \dots, T(\mathbf{v}_k)\}$ is a linearly independent set in W.

Corollary 6.23 Let $\dim V = \dim W = n$. Then a one-to-one linear transformation $T : V \to W$ maps a basis for V to a basis for W.

Example 6.71

Let $T : \mathbb{R}^2 \dashrightarrow \mathscr{P}_1$ be the linear transformation from Example 6.70, defined by

$$T\begin{bmatrix} a \\ b \end{bmatrix} = a + (a + b)x$$

Then, by Corollary 6.23, the standard basis $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$ for $\mathbb{R}^2$ is mapped to a basis $T(\mathcal{E}) = \{T(\mathbf{e}_1), T(\mathbf{e}_2)\}$ of $\mathscr{P}_1$.

We find that $T(\mathbf{e}_1) = T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 + x$ and $T(\mathbf{e}_2) = T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = x$

It follows that $\{1 + x, x\}$ is a basis for $\mathscr{P}_1$.

Theorem 6.24 A linear transformation $T : V \dashrightarrow W$ is invertible if and only if it is one-to-one and onto.

Isomorphisms of Vector Spaces

Definition A linear transformation $T : V \to W$ is called an ***isomorphism*** if it is one-to-one and onto. If V and W are two vector spaces such that there is an isomorphism from V to W, then we say that V is ***isomorphic*** to W and write $V \cong W$.

Example 6.72 Show that $\mathcal{P}_{n-1}$ and $\mathbb{R}^n$ are isomorphic.

If $p(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$ is in the kernel of T, th

$$\begin{bmatrix} a_0 \\ \vdots \\ a_{n-1} \end{bmatrix} = T(a_0 + a_1x + \cdots + a_{n-1}x^{n-1}) = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Hence, $a_0 = a_1 = \cdots = a_{n-1} = 0$, so $p(x) = 0$.

Therefore, $\ker(T) = \{0\}$, and T is one-to-one.

Since $\dim \mathcal{P}_{n-1} = \dim \mathbb{R}^n = n$, T is also onto, by Theorem 6.21.

Thus, T is an isomorphism, and $\mathcal{P}_{n-1} \cong \mathbb{R}^n$.

Example 6.73 Show that M_{mn} and $\mathbb{R}^{mn}$ are isomorphic.

Solution Once again, the coordinate mapping from M_{mn} to $\mathbb{R}^{mn}$ (as in Example 6.36) is an isomorphism. The details of the proof are left as an exercise.

Theorem 6.25 Let V and W be two finite-dimensional vector spaces (over the same field of scalars). Then V is isomorphic to W if and only if $\dim V = \dim W$.

Example 6.74 Show that $\mathbb{R}^n$ and $\mathscr{P}_n$ are not isomorphic.

Example 6.75

Let W be the vector space of all symmetric 2×2 matrices.

Show that W is isomorphic to $\mathbb{R}^3$.

Solution In Example 6.42, we showed that $\dim W = 3$. Hence, $\dim W = \dim \mathbb{R}^3$, so $W \cong \mathbb{R}^3$, by Theorem 6.25. (There is an obvious candidate for an isomorphism $T : W \to \mathbb{R}^3$. What is it?)

Exercises 6.5

2, 3, 6, 9, 13, 16, 19, 22, 24, 33