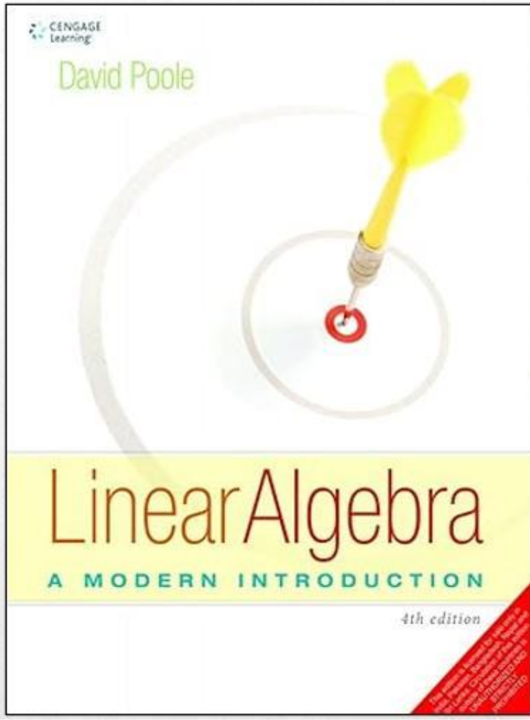

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線性代數 (二)

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6.4 Linear Transformations

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We encountered linear transformations in Section 3.6 in the context of matrix transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$. In this section, we extend this concept to linear transformations between arbitrary vector spaces.

Definition A ***linear transformation*** from a vector space V to a vector space W is a mapping $T : V \to W$ such that, for all $\mathbf{u}$ and $\mathbf{v}$ in V and for all scalars c,

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(c\mathbf{u}) = cT(\mathbf{u})$

$T : V \to W$ is a linear transformation if and only if

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_kT(\mathbf{v}_k)$$

for all $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in V and scalars $c_1, \ldots, c_k$.

Theorem 3.30 Let A be an $m \times n$ matrix. Then the matrix transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T_A(\mathbf{x}) = A\mathbf{x} \quad (\text{for } \mathbf{x} \text{ in } \mathbb{R}^n)$$

is a linear transformation.

Example 6.49 Every matrix transformation is a linear transformation. That is, if A is an $m \times n$ matrix, then the transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T_A(\mathbf{x}) = A\mathbf{x} \quad \text{for } \mathbf{x} \text{ in } \mathbb{R}^n$$

is a linear transformation. This is a restatement of Theorem 3.30.

Example 6.50 Define $T : M_{nn} \to M_{nn}$ by $T(A) = A^T$. Show that T is a linear transformation.

Example 6.51 Let D be the ***differential operator*** $D : \mathscr{D} \to \mathscr{F}$ defined by $D(f) = f'$. Show that D is a linear transformation.

Solution Let f and g be differentiable functions and let c be a scalar. Then, from calculus, we know that

$$D(f+g) = (f+g)' = f' + g' = D(f) + D(g)$$

and

$$D(cf) = (cf)' = cf' = cD(f)$$

Hence, D is a linear transformation.

Example 6.52 Define $S : \mathscr{C}[a, b] \to \mathbb{R}$ by $S(f) = \int_a^b f(x)\,dx$. Show that S is a linear transformation.

$$S(cf) = \int_a^b (cf)(x)\,dx = \int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx = cS(f)$$

It follows that S is linear.

Example 6.53

Show that none of the following transformations is linear:

(a) $T: M_{22} \to \mathbb{R}$ defined by $T(A) = \det A$

(b) $T: \mathbb{R} \to \mathbb{R}$ defined by $T(x) = 2^x$

(c) $T: \mathbb{R} \to \mathbb{R}$ defined by $T(x) = x + 1$

Solution give a specific counterexample

(a) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so

Example 6.53

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(c) $T: \mathbb{R} \to \mathbb{R}$ defined by $T(x) = x + 1$

Solution

(b) Let $x = 1$ and $y = 2$. Then

$$T(x + y) = T(3) = 2^3 = 8 \neq 6 = 2^1 + 2^2 = T(x) + T(y)$$

so T is not linear.

(c) Let $x = 1$ and $y = 2$. Then

$$T(x + y) = T(3) = 3 + 1 = 4 \neq 5 = (1 + 1) + (2 + 1) = T(x) + T(y)$$

Therefore, T is not linear.

Remark Example 6.53(c) shows that you need to be careful when you encounter the word "linear." As a *function*, $T(x) = x + 1$ is linear, since its graph is a straight line. However, it is not a *linear transformation* from the vector space $\mathbb{R}$ to itself, since it fails to satisfy the definition. (Which linear functions from $\mathbb{R}$ to $\mathbb{R}$ will also be linear transformations?)

There are two special linear transformations that deserve to be singled out.

Example 6.54

(a) For any vector spaces V and W, the transformation $T_0 : V \to W$ that maps every vector in V to the zero vector in W is called the ***zero transformation.*** That is,

$$T_0(\mathbf{v}) = \mathbf{0} \quad \text{for all } \mathbf{v} \text{ in } V$$

(b) For any vector space V, the transformation $I : V \to V$ that maps every vector in V to itself is called the ***identity transformation.*** That is,

$$I(\mathbf{v}) = \mathbf{v} \quad \text{for all } \mathbf{v} \text{ in } V$$

Properties of Linear Transformations

Theorem 6.14

Let $T : V \to W$ be a linear transformation. Then:

a. $T(\mathbf{0}) = \mathbf{0}$

b. $T(-\mathbf{v}) = -T(\mathbf{v})$ for all $\mathbf{v}$ in V.

c. $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$ for all $\mathbf{u}$ and $\mathbf{v}$ in V.

Example 6.55 Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathscr{P}_2$ such that

$$T\begin{bmatrix}1\\1\end{bmatrix} = 2 - 3x + x^2 \quad \text{and} \quad T\begin{bmatrix}2\\3\end{bmatrix} = 1 - x^2, \quad \text{Find } T\begin{bmatrix}-1\\2\end{bmatrix} \text{ and } T\begin{bmatrix}a\\b\end{bmatrix}.$$

Solution Since $\mathcal{B} = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^2$ (why?), every vector in $\mathbb{R}^2$ is in $\operatorname{span}(\mathcal{B})$. Solving

$$c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix}$$

so

$$\begin{aligned}
T\begin{bmatrix}a\\b\end{bmatrix} &= T\left((3a - 2b)\begin{bmatrix}1\\1\end{bmatrix} + (b - a)\begin{bmatrix}2\\3\end{bmatrix}\right)\\
&= (3a - 2b)T\begin{bmatrix}1\\1\end{bmatrix} + (b - a)T\begin{bmatrix}2\\3\end{bmatrix}\\
&= (3a - 2b)(2 - 3x + x^2) + (b - a)(1 - x^2)\\
&= (5a - 3b) + (-9a + 6b)x + (4a - 3b)x^2
\end{aligned}$$

(Note that by setting $a = -1$ and $b = 2$, we recover the solution $T\begin{bmatrix}-1\\2\end{bmatrix} = -11 + 21x - 10x^2$.)

Theorem 6.15 Let $T: V \to W$ be a linear transformation and let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a spanning set for V. Then $T(\mathcal{B}) = \{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ spans the range of T.

Composition of Linear Transformations

Definition If $T: U \to V$ and $S: V \to W$ are linear transformations, then the ***composition of S with T*** is the mapping $S \circ T$, defined by

$$(S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$$

where $\mathbf{u}$ is in U.

$S \circ T$ is read "S of T."

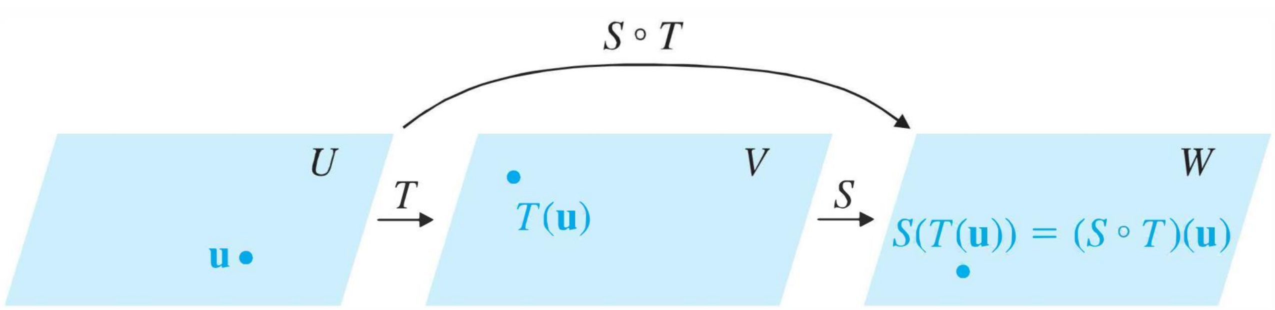

Figure 6.6

Observe that $S \circ T$ is a mapping from U to W (see Figure 6.6). Notice also that for the definition to make sense, the range of T must be contained in the domain of S.

Example 6.56

Let $T : \mathbb{R}^2 \to \mathscr{P}_1$ and $S : \mathscr{P}_1 \to \mathscr{P}_2$ be the linear transformations defined by

$$T\begin{bmatrix} a \\ b \end{bmatrix} = a + (a+b)x \quad \text{and} \quad S(p(x)) = xp(x)$$

Find $(S \circ T)\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $(S \circ T)\begin{bmatrix} a \\ b \end{bmatrix}$.

$$\begin{aligned}(S \circ T)\begin{bmatrix} a \\ b \end{bmatrix} &= S\left(T\begin{bmatrix} a \\ b \end{bmatrix}\right) = S(a + (a+b)x) = x(a + (a+b)x) \\ &= ax + (a+b)x^2\end{aligned}$$

Chapter 3 showed that the composition of two matrix transformations was another matrix transformation.

Theorem 6.16 If $T : U \to V$ and $S : V \to W$ are linear transformations, then $S \circ T : U \to W$ is a linear transformation.

Proof Let $\mathbf{u}$ and $\mathbf{v}$ be in U and let c be a scalar. Then

$$\begin{aligned} (S \circ T)(\mathbf{u} + \mathbf{v}) &= S(T(\mathbf{u} + \mathbf{v})) \\ &= S(T(\mathbf{u}) + T(\mathbf{v})) && \text{since } T \text{ is linear} \\ &= S(T(\mathbf{u})) + S(T(\mathbf{v})) && \text{since } S \text{ is linear} \\ &= (S \circ T)(\mathbf{u}) + (S \circ T)(\mathbf{v}) \end{aligned}$$

and

$$\begin{aligned} (S \circ T)(c\mathbf{u}) &= S(T(c\mathbf{u})) \\ &= S(cT(\mathbf{u})) && \text{since } T \text{ is linear} \\ &= cS(T(\mathbf{u})) && \text{since } S \text{ is linear} \\ &= c(S \circ T)(\mathbf{u}) \end{aligned}$$

Therefore, $S \circ T$ is a linear transformation.

Example 6.57 Let $S : U \to V$ and $T : V \to W$ be linear transformations and let $I : V \to V$ be the identity transformation. Then for every $\mathbf{v}$ in V, we have

$$(T \circ I)(\mathbf{v}) = T(I(\mathbf{v})) = T(\mathbf{v})$$

Since $T \circ I$ and T have the same value at every $\mathbf{v}$ in their domain, it follows that $T \circ I = T$. Similarly, $I \circ S = S$.

Remark The method of Example 6.57 is worth noting. Suppose we want to show that two linear transformations T_1 and T_2 (both from V to W) are equal. It suffices to show that $T_1(\mathbf{v}) = T_2(\mathbf{v})$ for every $\mathbf{v}$ in V.

Inverses of Linear Transformations

Definition A linear transformation $T : V \to W$ is ***invertible*** if there is a linear transformation $T' : W \to V$ such that

$$T' \circ T = I_V \quad \text{and} \quad T \circ T' = I_W$$

In this case, T' is called an ***inverse*** for T.

Example 6.58 Verify that the mappings $T:\mathbb{R}^2 \to \mathcal{P}_1$ and $T':\mathcal{P}_1 \to \mathbb{R}^2$ defined by

$$T\begin{bmatrix} a \\ b \end{bmatrix} = a + (a+b)x \quad \text{and} \quad T'(c+dx) = \begin{bmatrix} c \\ d-c \end{bmatrix}$$

are inverses.

Solution

$$(T' \circ T)\begin{bmatrix} a \\ b \end{bmatrix} = T'\left(T\begin{bmatrix} a \\ b \end{bmatrix}\right) = T'(a + (a+b)x) = \begin{bmatrix} a \\ (a+b) - a \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Hence, $T' \circ T = I_{\mathbb{R}^2}$ and $T \circ T' = I_{\mathcal{P}_1}$. Therefore, T and T' are inverses of each other.

Theorem 6.17 If T is an invertible linear transformation, then its inverse is unique.

Proof The proof is the same as that of Theorem 3.6, with products of matrices replaced by compositions of linear transformations. (You are asked to complete this proof in Exercise 31.)

Thanks to Theorem 6.17, if T is invertible, we can refer to *the* inverse of T. It will be denoted by T^{-1} (pronounced "T inverse").

In the next two sections, we will address the issue of determining when a given linear transformation is invertible and finding its inverse when it exists.

Exercises 6.4 2, 7, 9, 14, 19, 25, 29