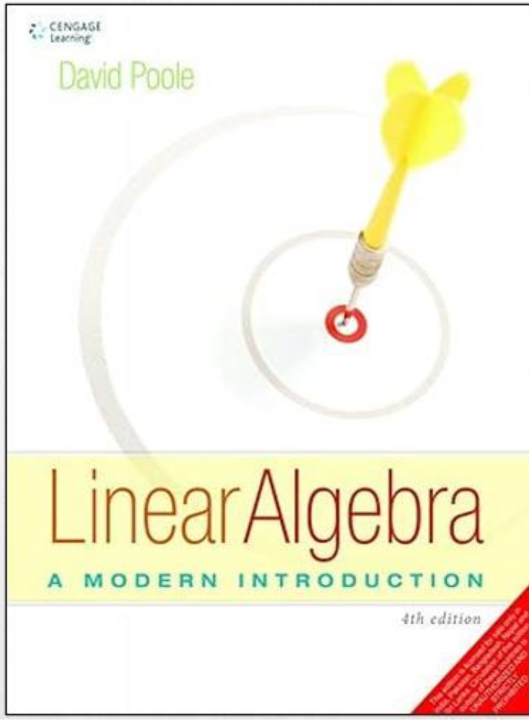

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線性代數 (二)

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6.3 Change of Basis

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Change-of-Basis Matrices

In many applications, a problem described using one coordinate system may be solved more easily by switching to a new coordinate system.

This switch is usually accomplished by performing a change of variables

In linear algebra, a basis provides
us with a coordinate system for a vector space, via the notion of coordinate vectors. Choosing the right basis will often greatly simplify a particular problem.

Figure 6.4(a) shows the coordinate system related to the basis $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2\}$, Figure 6.4(b) arises from the basis $\mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_2\}$, where

$$\mathbf{u}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

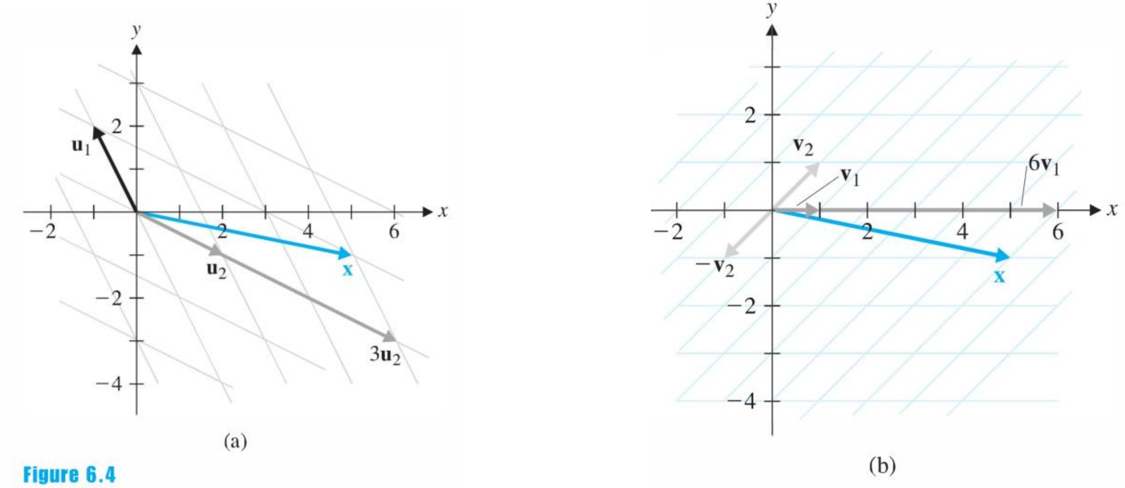


Figure 6.4

The same vector $\mathbf{x}$ is shown relative to each coordinate system. It is clear from the diagrams that the coordinate vectors of $\mathbf{x}$ with respect to $\mathcal{B}$ and $\mathcal{C}$ are

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \text{and} \quad [\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

It turns out that there is a direct connection between the two coordinate vectors. One way to find the relationship is to use $[\mathbf{x}]_{\mathcal{B}}$ to calculate

$$\mathbf{x} = \mathbf{u}_1 + 3\mathbf{u}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

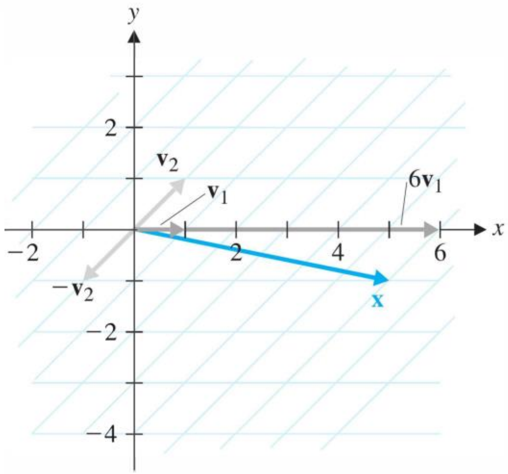
Then we can find $[\mathbf{x}]_{\mathcal{C}}$ by writing $\mathbf{x}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Example 6.45 Using the bases $\mathcal{B}$ and $\mathcal{C}$ above, find $[\mathbf{x}]_{\mathcal{C}}$, given that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Figure 6.4(a) shows the coordinate system related to the basis $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2\}$, Figure 6.4(b) arises from the basis $\mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_2\}$, where

$$\mathbf{u}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solution Since $\mathbf{x} = \mathbf{u}_1 + 3\mathbf{u}_2$, writing $\mathbf{u}_1$ and $\mathbf{u}_2$ in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$ will give us the required coordinates of $\mathbf{x}$ with respect to $\mathcal{C}$.



(b)

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$ in agreement with Figure 6.4(b).

Theorem 6.6 Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in V and let c be a scalar. Then

a. $[\mathbf{u} + \mathbf{v}]_{\mathcal{B}} = [\mathbf{u}]_{\mathcal{B}} + [\mathbf{v}]_{\mathcal{B}}$
b. $[c\mathbf{u}]_{\mathcal{B}} = c[\mathbf{u}]_{\mathcal{B}}$

Let's look at the calculations in Example 6.45 from a different point of view.

From $\mathbf{x} = \mathbf{u}_1 + 3\mathbf{u}_2$, we have

$$[\mathbf{x}]_{\mathcal{C}} = [\mathbf{u}_1 + 3\mathbf{u}_2]_{\mathcal{C}} = [\mathbf{u}_1]_{\mathcal{C}} + 3[\mathbf{u}_2]_{\mathcal{C}}$$

$$\mathbf{u}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = -3\mathbf{v}_1 + 2\mathbf{v}_2$$

$$\mathbf{u}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\mathbf{v}_1 - \mathbf{v}_2$$

Definition Let $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be bases for a vector space V. The $n \times n$ matrix whose columns are the coordinate vectors $[\mathbf{u}_1]_{\mathcal{C}}, \ldots, [\mathbf{u}_n]_{\mathcal{C}}$ of the vectors in $\mathcal{B}$ with respect to $\mathcal{C}$ is denoted by $P_{\mathcal{C} \leftarrow \mathcal{B}}$ and is called the ***change-of-basis matrix*** from $\mathcal{B}$ to $\mathcal{C}$. That is,

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = [\,[\mathbf{u}_1]_{\mathcal{C}}[\mathbf{u}_2]_{\mathcal{C}} \cdots [\mathbf{u}_n]_{\mathcal{C}}]$$

Theorem 6.12 Let $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be bases for a vector space V and let $P_{\mathcal{C}\leftarrow\mathcal{B}}$ be the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$. Then

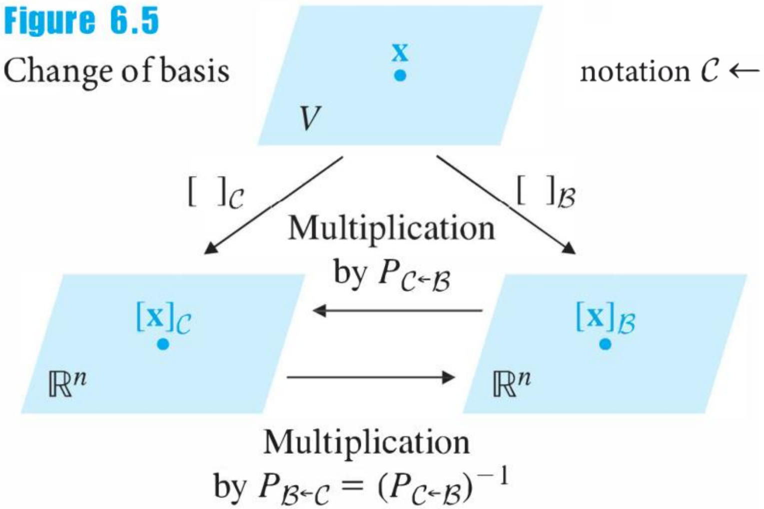
a. $P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$ for all $\mathbf{x}$ in V.
b. $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is the unique matrix P with the property that $P[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$ for all $\mathbf{x}$ in V.
c. $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is invertible and $(P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1} = P_{\mathcal{B}\leftarrow\mathcal{C}}$.

The columns of $P_{\mathcal{C}\leftarrow\mathcal{B}}$ are the coordinate vectors of one basis with respect to the other basis.

$P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$ is a linear combination of the columns of $P_{\mathcal{C}\leftarrow\mathcal{B}}$.

But since the result of this combination is $[\mathbf{x}]_{\mathcal{C}}$, the columns of $P_{\mathcal{C}\leftarrow\mathcal{B}}$ must themselves be coordinate vectors with respect to $\mathcal{C}$.

Figure 6.5
Change of basis



notation $\mathcal{C} \leftarrow \mathcal{B}$ as saying "$\mathcal{B}$ in terms of $\mathcal{C}$."

From $\quad \mathbf{x} = \mathbf{u}_1 + 3\mathbf{u}_2 \quad [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$[\mathbf{x}]_{\mathcal{C}} = [\mathbf{u}_1 + 3\mathbf{u}_2]_{\mathcal{C}} = [\mathbf{u}_1]_{\mathcal{C}} + 3[\mathbf{u}_2]_{\mathcal{C}}$$

Thus, $[\mathbf{x}]_{\mathcal{C}} = [[\mathbf{u}_1]_{\mathcal{C}}[\mathbf{u}_2]_{\mathcal{C}}]\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = P[\mathbf{x}]_{\mathcal{B}}$

Theorem 6.12 Let $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be bases for a vector space V and let $P_{\mathcal{C} \leftarrow \mathcal{B}}$ be the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$. Then

a. $P_{\mathcal{C} \leftarrow \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$ for all $\mathbf{x}$ in V.
b. $P_{\mathcal{C} \leftarrow \mathcal{B}}$ is the unique matrix P with the property that $P[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$ for all $\mathbf{x}$ in V.
c. $P_{\mathcal{C} \leftarrow \mathcal{B}}$ is invertible and $(P_{\mathcal{C} \leftarrow \mathcal{B}})^{-1} = P_{\mathcal{B} \leftarrow \mathcal{C}}$.

Theorem 6.12 Let $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be bases for a vector space V and let $P_{\mathcal{C}\leftarrow\mathcal{B}}$ be the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$. Then

a. $P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$ for all $\mathbf{x}$ in V.
b. $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is the unique matrix P with the property that $P[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$ for all $\mathbf{x}$ in V.
c. $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is invertible and $(P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1} = P_{\mathcal{B}\leftarrow\mathcal{C}}$.

(b) Suppose that P is an $n \times n$ matrix with the property that $P[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$ for all $\mathbf{x}$ in V.

Taking $\mathbf{x} = \mathbf{u}_i$, the ith basis vector in $\mathcal{B}$, we see that $[\mathbf{x}]_{\mathcal{B}} = [\mathbf{u}_i]_{\mathcal{B}} = \mathbf{e}_i$,

so the ith column of P is $\mathbf{p}_i = P\mathbf{e}_i = P[\mathbf{u}_i]_{\mathcal{B}} = [\mathbf{u}_i]_{\mathcal{C}}$

which is the ith column of $P_{\mathcal{C}\leftarrow\mathcal{B}}$, by definition. It follows that $P = P_{\mathcal{C}\leftarrow\mathcal{B}}$.

Theorem 6.12 Let $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be bases for a vector space V and let $P_{\mathcal{C}\leftarrow\mathcal{B}}$ be the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$. Then

a. $P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$ for all $\mathbf{x}$ in V.
b. $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is the unique matrix P with the property that $P[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$ for all $\mathbf{x}$ in V.
c. $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is invertible and $(P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1} = P_{\mathcal{B}\leftarrow\mathcal{C}}$.

(c) Since $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is linearly independent in V, the set $\{[\mathbf{u}_1]_{\mathcal{C}}, \dots, [\mathbf{u}_n]_{\mathcal{C}}\}$ is linearly independent in $\mathbb{R}^n$, by Theorem 6.7. Hence, $P_{\mathcal{C}\leftarrow\mathcal{B}} = [[\mathbf{u}_1]_{\mathcal{C}} \quad \cdots \quad [\mathbf{u}]_{\mathcal{C}}$ is invertible, by the Fundamental Theorem.

For all $\mathbf{x}$ in V, we have $P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$. Solving for $[\mathbf{x}]_{\mathcal{B}}$, we find that

$$[\mathbf{x}]_{\mathcal{B}} = (P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}[\mathbf{x}]_{\mathcal{C}}$$

for all $\mathbf{x}$ in V. Therefore, $(P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}$ is a matrix that changes bases from $\mathcal{C}$ to $\mathcal{B}$. Thus, by the uniqueness property (b), we must have $(P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1} = \mathcal{P}_{\mathcal{B}\leftarrow\mathcal{C}}$.

Example 6.46 Find the change-of-basis matrices $P_{\mathcal{C}\leftarrow\mathcal{B}}$ and $P_{\mathcal{B}\leftarrow\mathcal{C}}$ for the bases $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{C} = \{1 + x, x + x^2, 1 + x^2\}$ of $\mathscr{P}_2$. Then find the coordinate vector of $p(x) = 1 + 2x - x^2$ with respect to $\mathcal{C}$.

To find $P_{\mathcal{C}\leftarrow\mathcal{B}}$, use the fact that $P_{\mathcal{C}\leftarrow\mathcal{B}} = (P_{\mathcal{B}\leftarrow\mathcal{C}})^{-1}$, by Theorem 6.12(c). We find that

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = (P_{\mathcal{B}\leftarrow\mathcal{C}})^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

It now follows that

$$[p(x)]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[p(x)]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

Example 6.47 In M_{22}, let $\mathcal{B}$ be the basis $\{E_{11}, E_{21}, E_{12}, E_{22}\}$ and let $\mathcal{C}$ be the basis $\{A, B, C, D\}$, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Find the change-of-basis matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ and verify that $[X]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[X]_{\mathcal{B}}$ for $X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Solution 1 find the coordinate vectors of $\mathcal{B}$ with respect to $\mathcal{C}$.

This involves solving four linear combination problems of the form

$X = aA + bB + cC + dD$, where X is in $\mathcal{B}$ and we must find a, b, c, and d.

so $$P_{\mathcal{C}\leftarrow\mathcal{B}} = [[E_{11}]_{\mathcal{C}} \quad [E_{21}]_{\mathcal{C}} \quad [E_{12}]_{\mathcal{C}} \quad [E_{22}]_{\mathcal{C}}] = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 6.47 In M_{22}, let $\mathcal{B}$ be the basis $\{E_{11}, E_{21}, E_{12}, E_{22}\}$ and let $\mathcal{C}$ be the basis $\{A, B, C, D\}$, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Find the change-of-basis matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$ and verify that $[X]_{\mathcal{C}} = P_{\mathcal{C} \leftarrow \mathcal{B}}[X]_{\mathcal{B}}$ for $X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Solution

This is the coordinate vector with respect to $\mathcal{C}$ of the matrix

$$-A - B - C + 4D = -\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + 4\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = X \quad \text{as it should be.}$$

The Gauss-Jordan Method for Computing a Change-of-Basis Matrix

Theorem 6.13 Let $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be bases for a vector space V. Let $B = [[\mathbf{u}_1]_\mathcal{E} \ \cdots \ [\mathbf{u}_n]_\mathcal{E}]$ and $C = [[\mathbf{v}_1]_\mathcal{E} \ldots [\mathbf{v}_n]_\mathcal{E}]$, where $\mathcal{E}$ is any basis for V. Then row reduction applied to the $n \times 2n$ augmented matrix $[C \mid B]$ produces

$$[C \mid B] \rightarrow [I \mid P_{\mathcal{C} \leftarrow \mathcal{B}}]$$

Suppose $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are bases for a vector space V and $P_{\mathcal{C} \leftarrow \mathcal{B}}$ is the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$.

The ith column of P is $[\mathbf{u}_i]_\mathcal{C} = \begin{bmatrix} p_{1i} \\ \vdots \\ p_{ni} \end{bmatrix}$ so $\mathbf{u}_i = p_{1i}\mathbf{v}_1 + \cdots + p_{ni}\mathbf{v}_n$.

If $\mathcal{E}$ is a standard basis, $B = P_{\mathcal{E} \leftarrow \mathcal{B}}$ and $C = P_{\mathcal{E} \leftarrow \mathcal{C}}$.

If $\mathcal{E}$ is any basis for V, then

$$[\mathbf{u}_i]_\mathcal{E} = [p_{1i}\mathbf{v}_1 + \cdots + p_{ni}\mathbf{v}_n]_\mathcal{E} = p_{1i}[\mathbf{v}_1]_\mathcal{E} + \cdots + p_{ni}[\mathbf{v}_n]_\mathcal{E}$$

This can be rewritten in matrix form as

$$[[\mathbf{v}_1]_\mathcal{E} \ \cdots \ [\mathbf{v}_n]_\mathcal{E}] \begin{bmatrix} p_{1i} \\ \vdots \\ p_{ni} \end{bmatrix} = [\mathbf{u}_i]_\mathcal{E}$$

we can solve by applying Gauss-Jordan elimination to the augmented matrix

$$[[\mathbf{v}_1]_\mathcal{E} \ \cdots \ [\mathbf{v}_n]_\mathcal{E} \mid [\mathbf{u}_i]_\mathcal{E}]$$

which we can solve by applying Gauss-Jordan elimination to the augmented matrix

$$[[\mathbf{v}_1]_{\mathcal{E}} \ \cdots \ [\mathbf{v}_n]_{\mathcal{E}} \mid [\mathbf{u}_i]_{\mathcal{E}}]$$

There are n such systems of equations to be solved, one for each column of $P_{\mathcal{C}\leftarrow\mathcal{B}}$, but *the coefficient matrix* $[[\mathbf{v}_1]_{\mathcal{E}} \ \cdots \ [\mathbf{v}_n]_{\mathcal{E}}]$ *is the same in each case.*

Hence, we can solve all the systems simultaneously by row reducing the $n \times 2n$ augmented matrix

$$[[\mathbf{v}_1]_{\mathcal{E}} \ \cdots \ [\mathbf{v}_n]_{\mathcal{E}} \mid [\mathbf{u}_1]_{\mathcal{E}} \ \cdots \ [\mathbf{u}_n]_{\mathcal{E}}] = [C \mid B]$$

這裡的 B, C 矩陣就是 $P_{\mathcal{E}\leftarrow B}$ 和 $P_{\mathcal{E}\leftarrow C}$

Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent, so is $\{[\mathbf{v}_1]_{\mathcal{E}}, \ldots, [\mathbf{v}_n]_{\mathcal{E}}\}$, by Theorem 6.7.

Therefore, the matrix C whose columns are $[\mathbf{v}_1]_{\mathcal{E}}, \ldots, [\mathbf{v}_n]_{\mathcal{E}}$ has the $n \times n$ identity matrix I for its reduced row echelon form, by the Fundamental Theorem.

It follows that Gauss-Jordan elimination will necessarily produce

$$[C \mid B] \to [I \mid P] \qquad \text{where } P = P_{\mathcal{C}\leftarrow\mathcal{B}}.$$

If $\mathcal{E}$ is a standard basis,

$B = P_{\mathcal{E}\leftarrow\mathcal{B}}$ and $C = P_{\mathcal{E}\leftarrow\mathcal{C}}$.

Example 6.48

Rework Example 6.47 using the Gauss-Jordan method.

Example 6.47

In M_{22}, let $\mathcal{B}$ be the basis $\{E_{11}, E_{21}, E_{12}, E_{22}\}$ and let $\mathcal{C}$ be the basis $\{A, B, C, D\}$, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Find the change-of-basis matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ and verify that $[X]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[X]_{\mathcal{B}}$ for $X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

$\mathcal{E} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$

$B = [[E_{11}]_{\mathcal{E}}, [E_{21}]_{\mathcal{E}}, [E_{12}]_{\mathcal{E}}, [E_{22}]_{\mathcal{E}}]$

与基 $B = \{E_{11}, E_{21}, E_{12}, E_{22}\}$ 完全一致

Theorem 6.13

Let $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be bases for a vector space V. Let $B = [[\mathbf{u}_1]_{\mathcal{E}} \ \cdots \ [\mathbf{u}_n]_{\mathcal{E}}]$ and $C = [[\mathbf{v}_1]_{\mathcal{E}} \ldots [\mathbf{v}_n]_{\mathcal{E}}]$, where $\mathcal{E}$ is any basis for V. Then row reduction applied to the $n \times 2n$ augmented matrix $[C \mid B]$ produces

$$[C \mid B] \to [I \mid P_{\mathcal{C}\leftarrow\mathcal{B}}]$$

Solution Taking $\mathcal{E}$ to be the standard basis for M_{22}, we see that

$$B = P_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad C = P_{\mathcal{E}\leftarrow\mathcal{C}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If $\mathcal{E}$ is a standard basis,

$B = P_{\mathcal{E}\leftarrow\mathcal{B}}$ and $C = P_{\mathcal{E}\leftarrow\mathcal{C}}$.

Row reduction produces

$$[C \mid B] = \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

It follows that $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Exercises 6.3

3, 7, 10, 15