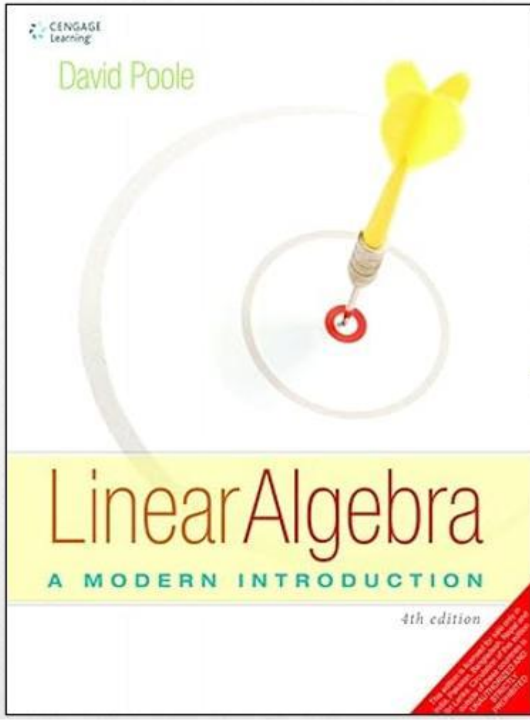

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線性代數 (二)

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6.2 Linear Independence, Basis, and Dimension

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extend the notions of linear independence, basis, and dimension to general vector spaces, generalizing the results of Sections 2.3 and 3.5.

2.3 Spanning Sets and Linear Independence
3.5 Subspaces, Basis, Dimension, and Rank

In most cases,
the proofs of the theorems carry over; we simply replace $\mathbb{R}^n$ by the vector space V.

Linear Independence

Definition A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in a vector space V is ***linearly dependent*** if there are scalars $c_1, c_2, \ldots, c_k$, *at least one of which is not zero*, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

A set of vectors that is not linearly dependent is said to be ***linearly independent***.

As in $\mathbb{R}^n$, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent in a vector space V if and only if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0} \quad \text{implies} \quad c_1 = 0, c_2 = 0, \ldots, c_k = 0$$

Theorem 6.4 A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in a vector space V is linearly dependent if and only if at least one of the vectors can be expressed as a linear combination of the others.

Example 6.22 In $\mathscr{P}_2$, the set $\{1 + x + x^2, 1 - x + 3x^2, 1 + 3x - x^2\}$ is linearly dependent, since

$$2(1 + x + x^2) - (1 - x + 3x^2) = 1 + 3x - x^2$$

Example 6.23 In M_{22}, let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

Then $A + B = C$, so the set $\{A, B, C\}$ is linearly dependent.

Example 6.24 In $\mathscr{F}$, the set $\{\sin^2 x, \cos^2 x, \cos 2x\}$ is linearly dependent, since

$$\cos 2x = \cos^2 x - \sin^2 x$$

Example 6.25 Show that the set $\{1, x, x^2, \ldots, x^n\}$ is linearly independent in $\mathcal{P}_n$.

Solution 2 assuming that

$$p(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n = 0$$

Since this is true for all x, we can substitute $x = 0$ to obtain $c_0 = 0$. This leaves

$$c_1x + c_2x^2 + \cdots + c_nx^n = 0$$

Taking derivatives, we obtain

$$c_1 + 2c_2x + 3c_3x^2 + \cdots + nc_nx^{n-1} = 0$$

and setting $x = 0$, we see that $c_1 = 0$.

Differentiating $2c_2x + 3c_3x^2 + \cdots + nc_nx^{n-1} = 0$

and setting $x = 0$, we find that $2c_2 = 0$, so $c_2 = 0$.

Continuing in this fashion, we find that $k!c_k = 0$ for $k = 0, \ldots, n$.
Therefore, $c_0 = c_1 = c_2 = \cdots = c_n = 0$, and $\{1, x, x^2, \ldots, x^n\}$ is linearly independent.

Example 6.26 In $\mathscr{P}_2$, determine whether the set $\{1 + x, x + x^2, 1 + x^2\}$ is linearly independent.

Solution Let c_1, c_2, and c_3 be scalars such that

$$c_1(1 + x) + c_2(x + x^2) + c_3(1 + x^2) = 0$$

Then

$$(c_1 + c_3) + (c_1 + c_2)x + (c_2 + c_3)x^2 = 0$$

This implies that

$$\begin{aligned} c_1 + c_3 &= 0 \\ c_1 + c_2 &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

the solution to which is $c_1 = c_2 = c_3 = 0$.

It follows that $\{1 + x, x + x^2, 1 + x^2\}$ is linearly independent.

Example 6.27 In $\mathscr{F}$, determine whether the set $\{\sin x, \cos x\}$ is linearly independent.

Solution

Suppose c and d are scalars such that

$$c \sin x + d \cos x = 0$$

Setting $x = 0$, we obtain $d = 0$,

setting $x = \pi/2$, we obtain $c = 0$.

Therefore, the set $\{\sin x, \cos x\}$ is linearly independent.

Although the definitions of linear dependence and independence are phrased in terms of *finite* sets of vectors, we can extend the concepts to *infinite* sets as follows:

A set S of vectors in a vector space V is ***linearly dependent*** if it contains finitely many linearly dependent vectors. A set of vectors that is not linearly dependent is said to be ***linearly independent***.

Example 6.28 In $\mathscr{P}$, show that $S = \{1, x, x^2, \ldots\}$ is linearly independent.

Solution Suppose there is a finite subset T of S that is linearly dependent.

Let x^m be the highest power of x in T
let x^n be the lowest power of x in T.

Then there are scalars $c_n, c_{n+1}, \ldots, c_m$, not all zero, such that

$$c_n x^n + c_{n+1} x^{n+1} + \cdots + c_m x^m = 0$$

But, by an argument similar to that used in Example 6.25,

this implies that $c_n = c_{n+1} = \cdots = c_m = 0$, which is a contradiction.

Hence, S cannot contain finitely many linearly dependent vectors,

so it is linearly independent.

Example 6.25 Show that the set $\{1, x, x^2, \ldots, x^n\}$ is linearly independent in $\mathscr{P}_n$.

Bases

The important concept of a basis now can be extended easily to arbitrary vector spaces.

Definition A subset $\mathcal{B}$ of a vector space V is a ***basis*** for V if

1. $\mathcal{B}$ spans V and
2. $\mathcal{B}$ is linearly independent.

Example 6.29

If $\mathbf{e}_i$ is the ith column of the $n \times n$ identity matrix, then $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis for $\mathbb{R}^n$, called the ***standard basis*** for $\mathbb{R}^n$.

Example 6.30

$\{1, x, x^2, \ldots, x^n\}$ is a basis for $\mathcal{P}_n$, called the ***standard basis*** for $\mathcal{P}_n$.

Example 6.31

The set $\mathcal{E} = \{E_{11}, \ldots, E_{1n}, E_{21}, \ldots, E_{2n}, E_{m1}, \ldots, E_{mn}\}$ is a basis for M_{mn}, where the matrices E_{ij} are as defined in Example 6.18. $\mathcal{E}$ is called the ***standard basis*** for M_{mn}.

We have already seen that $\mathcal{E}$ spans M_{mn}. It is easy to show that $\mathcal{E}$ is linearly independent. (Verify this!) Hence, $\mathcal{E}$ is a basis for M_{mn}.

Show that $M_{23} = \operatorname{span}(E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23})$, where

$$E_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad E_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad E_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(That is, E_{ij} is the matrix with a 1 in row i, column j and zeros elsewhere.)

Example 6.32 Show that $\mathcal{B} = \{1 + x, x + x^2, 1 + x^2\}$ is a basis for $\mathcal{P}_2$.

Solution We have already shown that $\mathcal{B}$ is linearly independent, in Example 6.26.

To show that $\mathcal{B}$ spans $\mathcal{P}_2$, let $a + bx + cx^2$ be an arbitrary polynomial in $\mathcal{P}_2$. We must show that there are scalars c_1, c_2, and c_3 such that

$$c_1(1 + x) + c_2(x + x^2) + c_3(1 + x^2) = a + bx + cx^2$$

or, equivalently, $$(c_1 + c_3) + (c_1 + c_2)x + (c_2 + c_3)x^2 = a + bx + cx^2$$

since the coefficient matrix $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ has rank 3 and, hence, is invertible.

Therefore, $\mathcal{B}$ is a basis for $\mathcal{P}_2$.

Example 6.33

Show that $\mathcal{B} = \{1, x, x^2, \ldots\}$ is a basis for $\mathscr{P}$.

Solution In Example 6.28, we saw that $\mathcal{B}$ is linearly independent. It also spans $\mathscr{P}$, since clearly every polynomial is a linear combination of (finitely many) powers of x.

Example 6.34

Find bases for the three vector spaces in Example 6.13:

(a) $W_1 = \left\{ \begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix} \right\}$

(a) Show that the set W of all vectors of the form

$$\begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix}$$ is a subspace of $\mathbb{R}^4$.

(a) Since

$$\begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

Example 6.34

Find bases for the three vector spaces in Example 6.13:

(b) $W_2 = \{a + bx - bx^2 + ax^3\}$

(b) Show that the set W of all polynomials of the form $a + bx - bx^2 + ax^3$ is subspace of $\mathscr{P}_3$.

(b) Since

$$a + bx - bx^2 + ax^3 = a(1 + x^3) + b(x - x^2)$$

Since $\{u(x), v(x)\}$ is clearly linearly independent, it is also a basis for W_2.

Example 6.34

Find bases for the three vector spaces in Example 6.13:

(c) $W_3 = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \right\}$

(c) Show that the set W of all matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ is a subspace of M_{22}.

(c) Since

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Since $\{U, V\}$ is clearly linearly in dependent,

it is also a basis for W_3.

Coordinates

Section 3.5 introduced the idea of the coordinates of a vector with respect to a basis for subspaces of $\mathbb{R}^n$. We now extend this concept to arbitrary vector spaces.

Theorem 6.5 Let V be a vector space and let $\mathcal{B}$ be a basis for V. For every vector $\mathbf{v}$ in V, there is exactly one way to write $\mathbf{v}$ as a linear combination of the basis vectors in $\mathcal{B}$.

The converse of Theorem 6.5 is also true. That is, if $\mathcal{B}$ is a set of vectors in a vector space V with the property that every vector in V can be written uniquely as a linear combination of the vectors in $\mathcal{B}$, then $\mathcal{B}$ is a basis for V (see Exercise 30). In this sense, the *unique representation property* characterizes a basis.

Definition Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V. Let $\mathbf{v}$ be a vector in V, and write $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$. Then $c_1, c_2, \ldots, c_n$ are called the ***coordinates of* v *with respect to* $\mathcal{B}$**, and the column vector

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

is called the ***coordinate vector of* v *with respect to* $\mathcal{B}$**.

Observe that if the basis $\mathcal{B}$ of V has n vectors, then $[\mathbf{v}]_{\mathcal{B}}$ is a (column) vector in $\mathbb{R}^n$.

Example 6.35 Find the coordinate vector $[p(x)]_{\mathcal{B}}$ of $p(x) = 2 - 3x + 5x^2$ with respect to the standard basis $\mathcal{B} = \{1, x, x^2\}$ of $\mathcal{P}_2$.

Solution The polynomial $p(x)$ is already a linear combination of 1, x, and x^2, so

$$[p(x)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$$

Remark The *order* in which the basis vectors appear in $\mathcal{B}$ affects the order of the entries in a coordinate vector. For example, in Example 6.35, assume that the standard basis vectors are ordered as $\mathcal{B}' = \{x^2, x, 1\}$. Then the coordinate vector of $p(x) = 2 - 3x + 5x^2$ with respect to $\mathcal{B}'$ is

$$[p(x)]_{\mathcal{B}'} = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$$

Example 6.36

Find the coordinate vector $[A]_{\mathcal{B}}$ of $A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$ with respect to the standard basis $\mathcal{B} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ of M_{22}.

Solution Since

$$A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 4\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= 2E_{11} - E_{12} + 4E_{21} + 3E_{22}$$

we have

$$[A]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 3 \end{bmatrix}$$

Example 6.37 Find the coordinate vector $[p(x)]_{\mathcal{B}}$ of $p(x) = 1 + 2x - x^2$ with respect to the basis $\mathcal{C} = \{1 + x, x + x^2, 1 + x^2\}$ of $\mathcal{P}_2$.

Solution We need to find c_1, c_2, and c_3 such that

$$c_1(1 + x) + c_2(x + x^2) + c_3(1 + x^2) = 1 + 2x - x^2$$

or, equivalently,

$$(c_1 + c_3) + (c_1 + c_2)x + (c_2 + c_3)x^2 = 1 + 2x - x^2$$

As in Example 6.32, this means we need to solve the system

$$\begin{aligned} c_1 + c_3 &= 1 \\ c_1 + c_2 &= 2 \\ c_2 + c_3 &= -1 \end{aligned}$$

whose solution is found to be $c_1 = 2$, $c_2 = 0$, $c_3 = -1$. Therefore,

$$[p(x)]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

Theorem 6.6 Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in V and let c be a scalar. Then

a. $[\mathbf{u} + \mathbf{v}]_{\mathcal{B}} = [\mathbf{u}]_{\mathcal{B}} + [\mathbf{v}]_{\mathcal{B}}$
b. $[c\mathbf{u}]_{\mathcal{B}} = c[\mathbf{u}]_{\mathcal{B}}$

Proof We begin by writing $\mathbf{u}$ and $\mathbf{v}$ in terms of the basis vectors—say, as

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \quad \text{and} \quad \mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_n\mathbf{v}_n$$

$$\mathbf{u} + \mathbf{v} = (c_1 + d_1)\mathbf{v}_1 + (c_2 + d_2)\mathbf{v}_2 + \cdots + (c_n + d_n)\mathbf{v}_n$$

$$c\mathbf{u} = (cc_1)\mathbf{v}_1 + (cc_2)\mathbf{v}_2 + \cdots + (cc_n)\mathbf{v}_n$$

Theorem 6.7 Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V and let $\mathbf{u}_1, \ldots, \mathbf{u}_k$ be vectors in V. Then $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is linearly independent in V if and only if $\{[\mathbf{u}_1]_{\mathcal{B}}, \ldots, [\mathbf{u}_k]_{\mathcal{B}}\}$ is linearly independent in $\mathbb{R}^n$.

Dimension

The definition of dimension is the same for a vector space as for a subspace of $\mathbb{R}^n$—the number of vectors in a basis for the space.

Since a vector space can have more than one

basis, we need to show that this definition makes sense;

Theorem 6.8 Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V.

a. Any set of more than n vectors in V must be linearly dependent.
b. Any set of fewer than n vectors in V cannot span V.

Theorem 6.9 **The Basis Theorem**

If a vector space V has a basis with n vectors, then every basis for V has exactly n vectors.

Definition A vector space V is called ***finite-dimensional*** if it has a basis consisting of finitely many vectors. The ***dimension*** of V, denoted by dim V, is the number of vectors in a basis for V. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero. A vector space that has no finite basis is called ***infinite-dimensional.***

Table 6.1

dim V	V
3	$\mathbb{R}^3$
2	Plane through the origin
1	Line through the origin
0	$\{\mathbf{0}\}$

Example 6.39

The standard basis for $\mathscr{P}_n$ contains $n + 1$ vectors (see Example 6.30), so $\dim \mathscr{P}_n = n + 1$.

Example 6.40

The standard basis for M_{mn} contains mn vectors (see Example 6.31), so $\dim M_{mn} = mn$.

Example 6.41

Both $\mathscr{P}$ and $\mathscr{F}$ are infinite-dimensional, since they each contain the infinite linearly independent set $\{1, x, x^2, \ldots\}$ (see Exercise 44).

Theorem 6.10

Let V be a vector space with $\dim V = n$. Then:

a. Any linearly independent set in V contains at most n vectors.
b. Any spanning set for V contains at least n vectors.
c. Any linearly independent set of exactly n vectors in V is a basis for V.
d. Any spanning set for V consisting of exactly n vectors is a basis for V.
e. Any linearly independent set in V can be extended to a basis for V.
f. Any spanning set for V can be reduced to a basis for V.

Example 6.42 Find the dimension of the vector space W of symmetric 2×2 matrices (see Example 6.10).

Solution A symmetric 2×2 matrix is of the form

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

so W is spanned by the set

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

from which it immediately follows that $a = b = c = 0$.

Hence, S is linearly independent and is,

therefore, a basis for W. We conclude that $\dim W = 3$.

Example 6.43

In each case, determine whether S is a basis for V.

(a) $V = \mathscr{P}_2,\ S = \{1 + x, 2 - x + x^2, 3x - 2x^2, -1 + 3x + x^2\}$

(b) $V = M_{22},\ S = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \right\}$

(c) $V = \mathscr{P}_2,\ S = \{1 + x, x + x^2, 1 + x^2\}$

Solution (a) Since $\dim(\mathscr{P}_2) = 3$ and S contains four vectors, S is linearly dependent, by Theorem 6.10(a). Hence, S is not a basis for $\mathscr{P}_2$.

(b) Since $\dim(M_{22}) = 4$ and S contains three vectors, S cannot span M_{22}, by Theorem 6.10(b). Hence, S is not a basis for M_{22}.

(c) Since $\dim(\mathscr{P}_2) = 3$ and S contains three vectors, S will be a basis for $\mathscr{P}_2$ if it is linearly independent or if it spans $\mathscr{P}_2$, by Theorem 6.10(c) or (d).

It is easier to show that S is linearly independent; Example 6.26. Therefore, S is a basis for $\mathscr{P}_2$.

Theorem 6.10 Let V be a vector space with $\dim V = n$. Then:

a. Any linearly independent set in V contains at most n vectors.
b. Any spanning set for V contains at least n vectors.
c. Any linearly independent set of exactly n vectors in V is a basis for V.
d. Any spanning set for V consisting of exactly n vectors is a basis for V.
e. Any linearly independent set in V can be extended to a basis for V.
f. Any spanning set for V can be reduced to a basis for V.

Exercises 6.2

2, 7, 12, 18, 23, 28, 36, 37