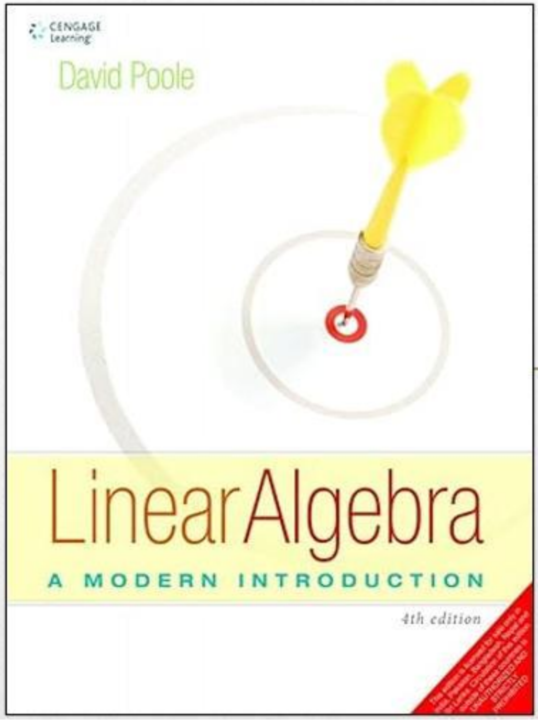

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線性代數 (二)

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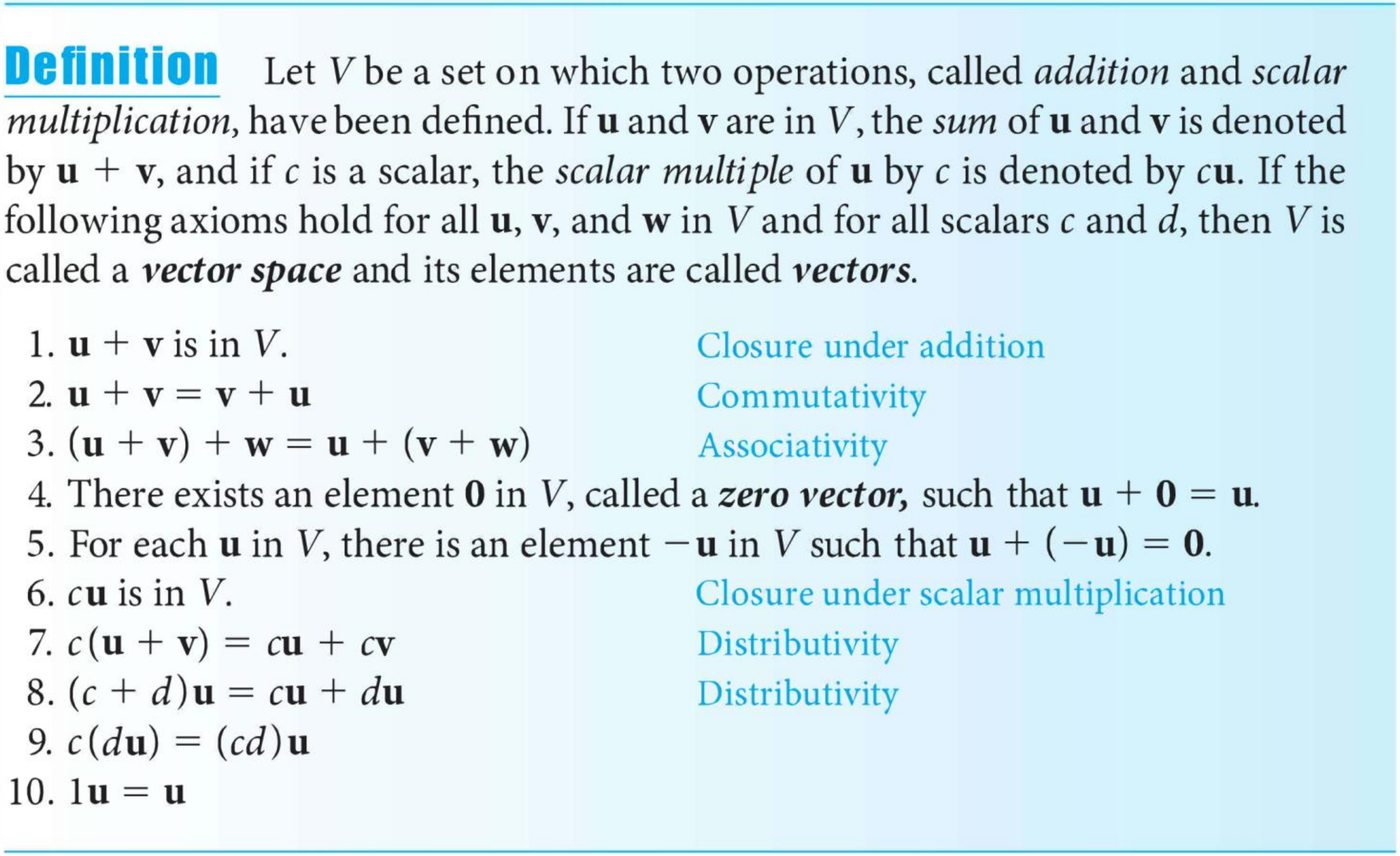
6.1 Vector Spaces and Subspaces

https://hmwu.idv.tw

In Chapters 1 and 3, we saw that the algebra of vectors and the algebra of matrices are similar in many respects.

In this section,
we use these properties to define generalized "vectors" that arise in a wide variety of examples.

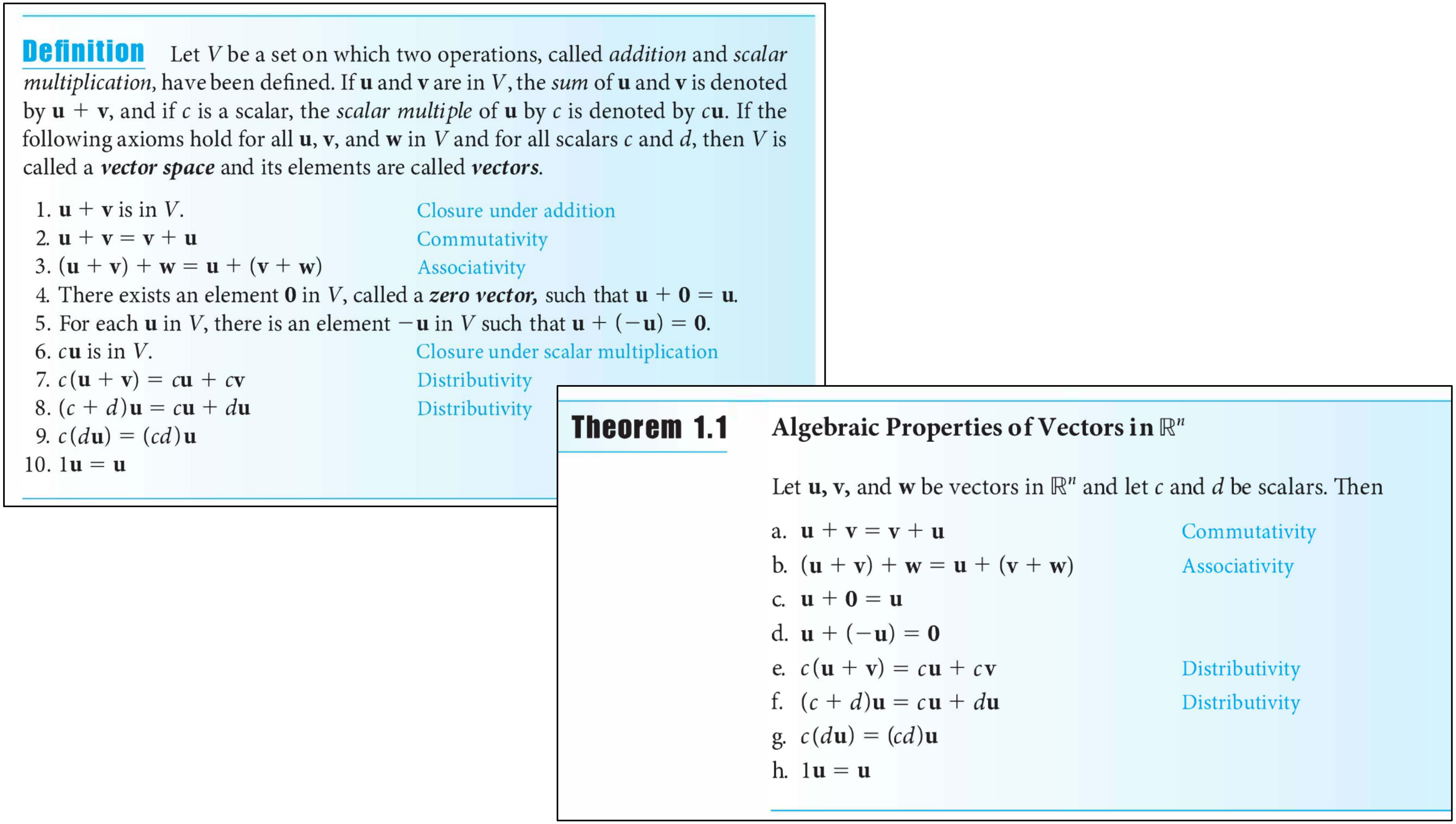


Definition Let V be a set on which two operations, called *addition* and *scalar multiplication*, have been defined. If $\mathbf{u}$ and $\mathbf{v}$ are in V, the *sum* of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} + \mathbf{v}$, and if c is a scalar, the *scalar multiple* of $\mathbf{u}$ by c is denoted by $c\mathbf{u}$. If the following axioms hold for all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d, then V is called a ***vector space*** and its elements are called ***vectors***.

1. $\mathbf{u} + \mathbf{v}$ is in V. Closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ Commutativity
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ Associativity
4. There exists an element $\mathbf{0}$ in V, called a ***zero vector,*** such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is an element $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. $c\mathbf{u}$ is in V. Closure under scalar multiplication
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ Distributivity
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ Distributivity
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

Example 6.1

For any $n \geq 1$, $\mathbb{R}^n$ is a vector space with the usual operations of addition and scalar multiplication. Axioms 1 and 6 follow from the definitions of these operations, and the remaining axioms follow from Theorem 1.1.



Definition Let V be a set on which two operations, called *addition* and *scalar multiplication*, have been defined. If $\mathbf{u}$ and $\mathbf{v}$ are in V, the *sum* of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} + \mathbf{v}$, and if c is a scalar, the *scalar multiple* of $\mathbf{u}$ by c is denoted by $c\mathbf{u}$. If the following axioms hold for all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d, then V is called a ***vector space*** and its elements are called ***vectors***.

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2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ Commutativity
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5. For each $\mathbf{u}$ in V, there is an element $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. $c\mathbf{u}$ is in V. Closure under scalar multiplication
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ Distributivity
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ Distributivity
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

Theorem 1.1 **Algebraic Properties of Vectors in $\mathbb{R}^n$**

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$ and let c and d be scalars. Then

a. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ Commutativity
b. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ Associativity
c. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
d. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
e. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ Distributivity
f. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ Distributivity
g. $c(d\mathbf{u}) = (cd)\mathbf{u}$
h. $1\mathbf{u} = \mathbf{u}$

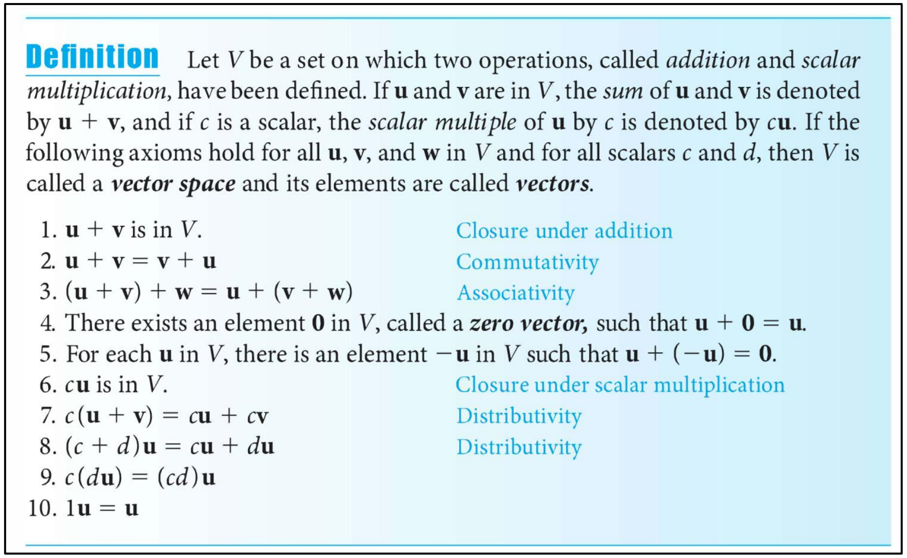
Example 6.2

The set of all 2 × 3 matrices is a vector space with the usual operations of matrix addition and matrix scalar multiplication. Here the "vectors" are actually matrices.

We know that the sum of two 2 × 3 matrices is also a 2 × 3 matrix and that multiplying a 2 × 3 matrix by a scalar gives another 2 × 3 matrix; hence, we have closure.

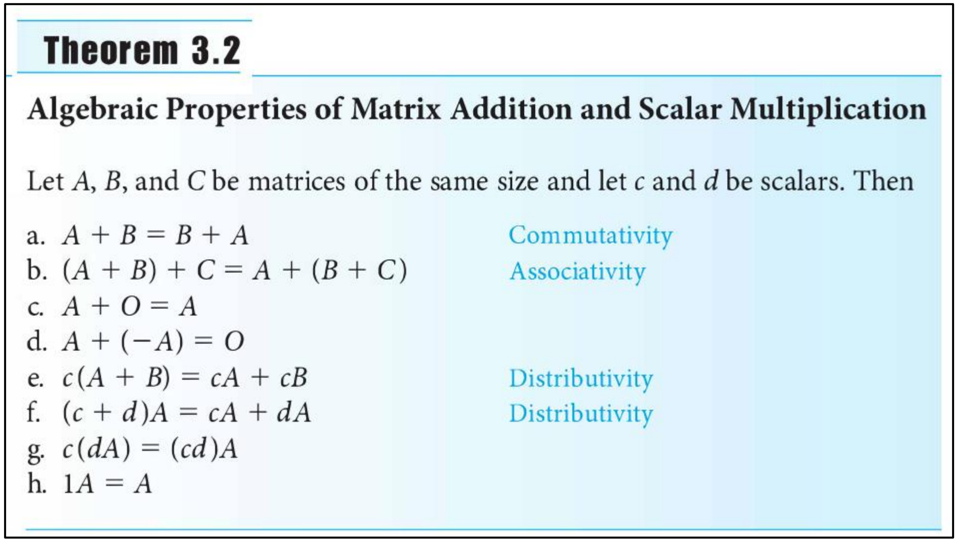
The remaining axioms follow from Theorem 3.2. In particular, the zero vector **0** is the 2 × 3 zero matrix, and the negative of a 2 × 3 matrix A is just the 2 × 3 matrix $-A$.

There is nothing special about 2 × 3 matrices. For any positive integers m and n, the set of all $m \times n$ matrices forms a vector space with the usual operations of matrix addition and matrix scalar multiplication. This vector space is denoted M_{mn}.



Definition Let V be a set on which two operations, called *addition* and *scalar multiplication*, have been defined. If $\mathbf{u}$ and $\mathbf{v}$ are in V, the *sum* of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} + \mathbf{v}$, and if c is a scalar, the *scalar multiple* of $\mathbf{u}$ by c is denoted by $c\mathbf{u}$. If the following axioms hold for all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d, then V is called a ***vector space*** and its elements are called ***vectors***.

1. $\mathbf{u} + \mathbf{v}$ is in V.	Closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutativity
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	Associativity
4. There exists an element $\mathbf{0}$ in V, called a ***zero vector***, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.	
5. For each $\mathbf{u}$ in V, there is an element $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.	
6. $c\mathbf{u}$ is in V.	Closure under scalar multiplication
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	Distributivity
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$	Distributivity
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$	
10. $1\mathbf{u} = \mathbf{u}$	



Theorem 3.2

Algebraic Properties of Matrix Addition and Scalar Multiplication

Let A, B, and C be matrices of the same size and let c and d be scalars. Then

a. $A + B = B + A$	Commutativity
b. $(A + B) + C = A + (B + C)$	Associativity
c. $A + O = A$	
d. $A + (-A) = O$	
e. $c(A + B) = cA + cB$	Distributivity
f. $(c + d)A = cA + dA$	Distributivity
g. $c(dA) = (cd)A$	
h. $1A = A$	

Example 6.3

Let $\mathscr{P}_2$ denote the set of all polynomials of degree 2 or less with real coefficients. Define addition and scalar multiplication in the usual way. (See Appendix D.) If

Appendix D*

Polynomials

A ***polynomial*** is a function p of a single variable x that can be written in the form

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \qquad (1)$$

where $a_0, a_1, \ldots, a_n$ are constants ($a_n \neq 0$), called the ***coefficients*** of p.

If $p(x) = a_0 + a_1x + a_2x^2$ and $q(x) = b_0 + b_1x + b_2x^2$ are in $\mathscr{P}_2$, then $p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$

has degree at most 2 and so is in $\mathscr{P}_2$.

It is now easy to verify the remaining axioms.

check axiom 2 and leave the others for Exercise 12.

With $p(x)$ and $q(x)$ as above, we have

$$\begin{aligned} p(x) + q(x) &= (a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2) \\ &= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 \\ &= (b_0 + a_0) + (b_1 + a_1)x + (b_2 + a_2)x^2 \\ &= (b_0 + b_1x + b_2x^2) + (a_0 + a_1x + a_2x^2) \\ &= q(x) + p(x) \end{aligned}$$

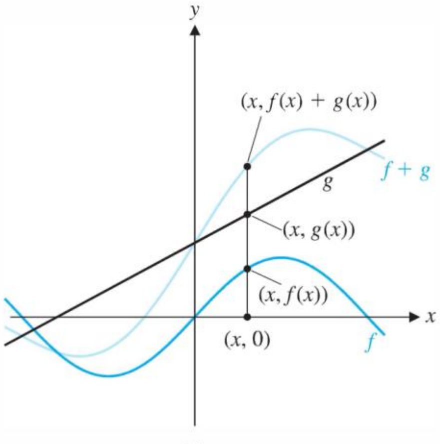
where the third equality follows from the fact that addition of real numbers is commutative.

In general, for any fixed $n \geq 0$, the set $\mathscr{P}_n$ of all polynomials of degree less than or equal to n is a vector space, as is the set $\mathscr{P}$ of *all* polynomials.

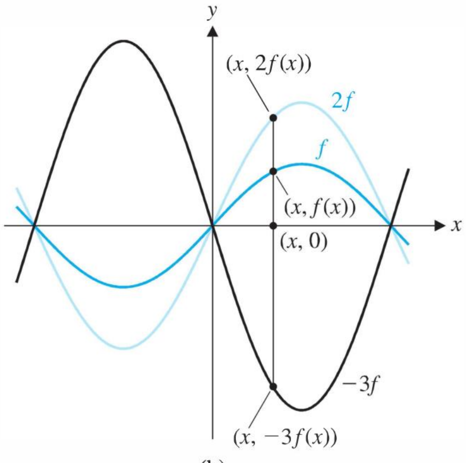
Example 6.4 Let $\mathscr{F}$ denote the set of all real-valued functions defined on the real line.

If f and g are two such functions and c is a scalar, then $f + g$ and cf are defined by

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (cf)(x) = cf(x)$$



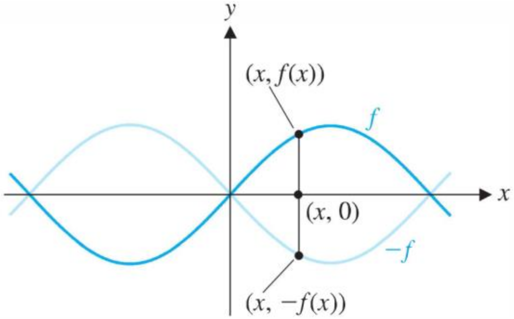
(a)



(b)

Axioms 1 and 6 are obviously true.

Verification of the remaining axioms is left as Exercise 13. Thus, $\mathscr{F}$ is a vector space.



(c)

Example 6.5

The set $\mathbb{Z}$ of integers with the usual operations is *not* a vector space. To demonstrate this, it is enough to find that *one* of the ten axioms fails and to give a specific instance in which it fails (a *counterexample*). In this case, we find that we do not have closure under scalar multiplication. For example, the multiple of the integer 2 by the scalar $\frac{1}{3}$ is $(\frac{1}{3})(2) = \frac{2}{3}$, which is not an integer. Thus, it is not true that cx is in $\mathbb{Z}$ for *every* x in $\mathbb{Z}$ and *every* scalar c (i.e., axiom 6 fails).

Example 6.6

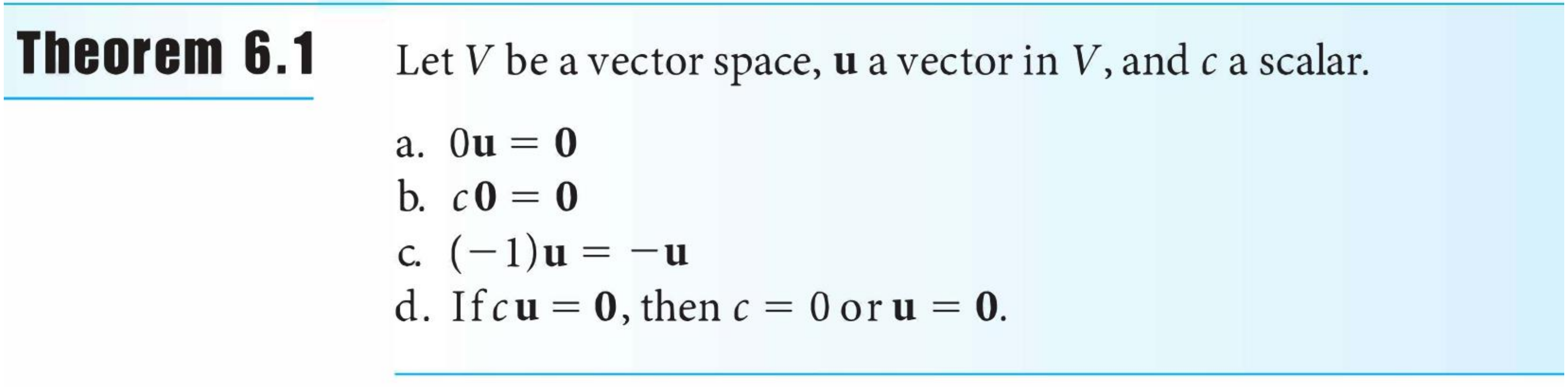
Let $V = \mathbb{R}^2$ with the usual definition of addition but the following definition of scalar multiplication:

$$c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ 0 \end{bmatrix}$$

Then, for example,

$$1\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

so axiom 10 fails. [In fact, the other nine axioms are all true (check this), but we do not need to look into them, because V has already failed to be a vector space. This example shows the value of looking ahead, rather than working through the list of axioms in the order in which they have been given.]



Theorem 6.1 Let V be a vector space, $\mathbf{u}$ a vector in V, and c a scalar.

a. $0\mathbf{u} = \mathbf{0}$
b. $c\mathbf{0} = \mathbf{0}$
c. $(-1)\mathbf{u} = -\mathbf{u}$
d. If $c\mathbf{u} = \mathbf{0}$, then $c = 0$ or $\mathbf{u} = \mathbf{0}$.

Subspaces

We have seen that, in $\mathbb{R}^n$, it is possible for one vector space to sit inside another one, giving rise to the notion of a subspace. For example, a plane through the origin is a subspace of $\mathbb{R}^3$. We now extend this concept to general vector spaces.

Definition A subset W of a vector space V is called a ***subspace*** of V if W is itself a vector space with the same scalars, addition, and scalar multiplication as V.

Theorem 6.2 Let V be a vector space and let W be a nonempty subset of V. Then W is a subspace of V if and only if the following conditions hold:

a. If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u} + \mathbf{v}$ is in W.
b. If $\mathbf{u}$ is in W and c is a scalar, then $c\mathbf{u}$ is in W.

Remark Since Theorem 6.2 generalizes the notion of a subspace from the context of $\mathbb{R}^n$ to general vector spaces, all of the subspaces of $\mathbb{R}^n$ that we encountered in Chapter 3 are subspaces of $\mathbb{R}^n$ in the current context. In particular, lines and planes through the origin are subspaces of $\mathbb{R}^3$.

Example 6.9 We have already shown that the set $\mathscr{P}_n$ of all polynomials with degree at most n is a vector space. Hence, $\mathscr{P}_n$ is a subspace of the vector space $\mathscr{P}$ of *all* polynomials. ▲

Example 6.10 Let W be the set of symmetric $n \times n$ matrices. Show that W is a subspace of M_{nn}.

Solution W is nonempty, so we need only check conditions (a) and (b) in Theorem 6.2.

We have shown that W is closed under addition and scalar multiplication. Therefore, it is a subspace of M_{nn}, by Theorem 6.2.

Example 6.11

Let $\mathscr{C}$ be the set of all continuous real-valued functions defined on $\mathbb{R}$ and let $\mathscr{D}$ be the set of all differentiable real-valued functions defined on $\mathbb{R}$. Show that $\mathscr{C}$ and $\mathscr{D}$ are subspaces of $\mathscr{F}$, the vector space of all real-valued functions defined on $\mathbb{R}$.

Solution From calculus, we know that if f and g are continuous functions and c is a scalar, then $f + g$ and cf are also continuous. Hence, $\mathscr{C}$ is closed under addition and scalar multiplication and so is a subspace of $\mathscr{F}$.

So $\mathscr{D}$ is also closed under addition and scalar multiplication, making it a subspace of $\mathscr{F}$.

Example 6.13

(a) Show that the set W of all vectors of the form

$$\begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix}$$

is a subspace of $\mathbb{R}^4$.

Let $\mathbf{u}$ and $\mathbf{v}$ be in W—say,

$$\mathbf{u} = \begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} c \\ d \\ -d \\ c \end{bmatrix}$$

Then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} a + c \\ b + d \\ -b - d \\ a + c \end{bmatrix} = \begin{bmatrix} a + c \\ b + d \\ -(b + d) \\ a + c \end{bmatrix}$$

so $\mathbf{u} + \mathbf{v}$ is also in W (because it has the right *form*).

Thus, W is a nonempty subset of $\mathbb{R}^4$ that is closed under addition and scalar multiplication.

Therefore, W is a subspace of $\mathbb{R}^4$, by Theorem 6.2.

Example 6.13

(b) Show that the set W of all polynomials of the form $a + bx - bx^2 + ax^3$ is a subspace of $\mathscr{P}_3$.

(b) W is nonempty because it contains the zero polynomial. (Take $a = b = 0$.)

Let $p(x)$ and $q(x)$ be in W—say,

$$p(x) = a + bx - bx^2 + ax^3 \quad \text{and} \quad q(x) = c + dx - dx^2 + cx^3$$

Then $$p(x) + q(x) = (a + c) + (b + d)x - (b + d)x^2 + (a + c)x^3$$

so $p(x) + q(x)$ is also in W (because it has the right *form*).

Thus, W is a nonempty subset of $\mathscr{P}_3$ that is closed under addition and scalar multiplication. Therefore, W is a subspace of $\mathscr{P}_3$ by Theorem 6.2.

Example 6.13

(c) Show that the set W of all matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ is a subspace of M_{22}.

(c) W is nonempty because it contains the zero matrix O. (Take $a = b = 0$.)

Similarly, if k is a scalar, then $kA = \begin{bmatrix} ka & kb \\ -kb & ka \end{bmatrix}$ so kA is in W.

Thus, W is a nonempty subset of M_{22} that is closed under addition and scalar multiplication. Therefore, W is a subspace of M_{22}, by Theorem 6.2.

Example 6.14

If V is a vector space, then V is clearly a subspace of itself. The set $\{\mathbf{0}\}$, consisting of only the zero vector, is also a subspace of V, called the ***zero subspace.*** To show this, we simply note that the two closure conditions of Theorem 6.2 are satisfied:

$$\mathbf{0} + \mathbf{0} = \mathbf{0} \quad \text{and} \quad c\mathbf{0} = \mathbf{0} \quad \text{for any scalar } c$$

The subspaces $\{\mathbf{0}\}$ and V are called the ***trivial subspaces*** of V.

Example 6.15

Let W be the set of all 2×2 matrices of the form

$$\begin{bmatrix} a & a+1 \\ 0 & b \end{bmatrix} \quad \text{Is } W \text{ a subspace of } M_{22}?$$

Solution Each matrix in W has the property that its (1, 2) entry is one more than its (1, 1) entry. Since the zero matrix

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

does not have this property, it is not in W.

Hence, W is not a subspace of M_{22}.

Example 6.16

Let W be the set of all 2×2 matrices with determinant equal to 0. Is W a subspace of M_{22}? (Since $\det O = 0$, the zero matrix is in W, so the method of Example 6.15 is of no use to us.)

Thus, W is not closed under addition and so is not a subspace of M_{22}.

Spanning Sets

The notion of a spanning set of vectors carries over easily from $\mathbb{R}^n$ to general vector spaces.

Definition If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is called the ***span*** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and is denoted by $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ or $\operatorname{span}(S)$. If $V = \operatorname{span}(S)$, then S is called a ***spanning set*** for V and V is said to be ***spanned*** by S.

Example 6.17

Show that the polynomials $1, x,$ and x^2 span $\mathcal{P}_2$.

Solution By its very definition, a polynomial $p(x) = a + bx + cx^2$ is a linear combination of 1, x, and x^2. Therefore, $\mathcal{P}_2 = \operatorname{span}(1, x, x^2)$.

Example 6.18 Show that $M_{23} = \operatorname{span}(E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23})$, where

$$E_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad E_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad E_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution We need only observe that

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = a_{11}E_{11} + a_{12}E_{12} + a_{13}E_{13} + a_{21}E_{21} + a_{22}E_{22} + a_{23}E_{23}$$

Example 6.19 In $\mathscr{P}_2$, determine whether $r(x) = 1 - 4x + 6x^2$ is in span$(p(x), q(x))$, where

$$p(x) = 1 - x + x^2 \quad \text{and} \quad q(x) = 2 + x - 3x^2$$

Solution We are looking for scalars c and d such that $cp(x) + dq(x) = r(x)$. This means that

$$c(1 - x + x^2) + d(2 + x - 3x^2) = 1 - 4x + 6x^2$$

Example 6.20 In $\mathscr{F}$, determine whether $\sin 2x$ is in $\operatorname{span}(\sin x, \cos x)$.

Solution We set $c \sin x + d \cos x = \sin 2x$ and try to determine c and d so that this equation is true.

But this implies that $\sin 2x = 0(\sin x) + 0(\cos x) = 0$ for all x, which is absurd, since $\sin 2x$ is not the zero function.

We conclude that $\sin 2x$ is not in $\operatorname{span}(\sin x, \cos x)$.

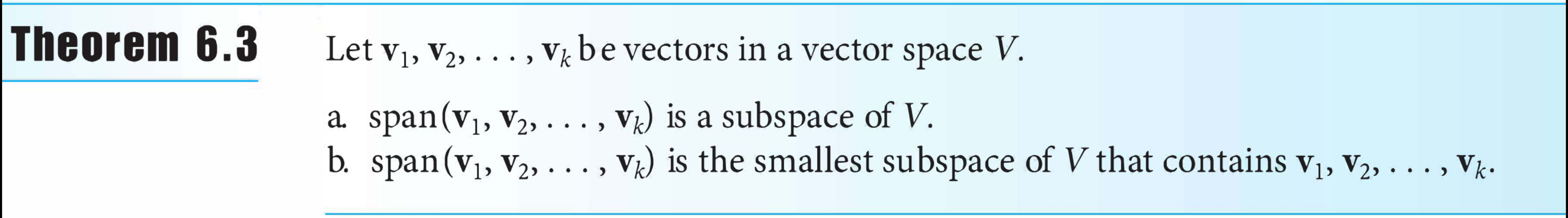
Example 6.21 In M_{22}, describe the span of $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Solution Every linear combination of A, B, and C is of the form

$$cA + dB + eC = c\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} c + d & c + e \\ c + e & d \end{bmatrix}$$

This matrix is symmetric, so span(A, B, C) is contained within the subspace of symmetric 2×2 matrices.

In fact, we have equality; *every* symmetric 2×2 matrix is in span(A, B, C).



Theorem 6.3 Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in a vector space V.

a. $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ is a subspace of V.
b. $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ is the smallest subspace of V that contains $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.

Exercises 6.1

5, 6, 9, 26, 27, 32, 43, 53, 60, 61