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線性代數 (二)

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5.4



Orthogonal Diagonalization of Symmetric Matrices

<https://hmwu.idv.tw>

Definition A square matrix A is *orthogonally diagonalizable* if there exists an orthogonal matrix Q and a diagonal matrix D such that $Q^T A Q = D$.

Example 5.16

If possible, diagonalize the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$.

Solution The characteristic polynomial of A is $\lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2)$,
eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 2$.

the corresponding eigenvectors, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Q is an *orthogonal* matrix, Therefore, $Q^{-1} = Q^T$, and we have $Q^T A Q = D$.

Theorem 5.17

If A is orthogonally diagonalizable, then A is symmetric.

Proof If A is orthogonally diagonalizable, then there exists an orthogonal matrix Q and a diagonal matrix D such that $Q^T A Q = D$.

since every diagonal matrix is symmetric. Hence, A is symmetric.

Remark

Theorem 5.17 shows that the orthogonally diagonalizable matrices are all to be found *among* the symmetric matrices.

It does *not* say that every symmetric matrix must be orthogonally diagonalizable.

Theorem 5.18

If A is a real symmetric matrix, then the eigenvalues of A are real.

Theorem 5.19

If A is a symmetric matrix, then any two eigenvectors corresponding to distinct eigenvalues of A are orthogonal.

Example 5.17

Verify the result of Theorem 5.19 for

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$E_4 = \text{span} \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \quad \text{and} \quad E_1 = \text{span} \left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right)$$

Theorem 5.20 The Spectral Theorem

Let A be an $n \times n$ real matrix. Then A is symmetric if and only if it is orthogonally diagonalizable.

Example 5.18

Example 5.17.

Orthogonally diagonalize the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$E_4 = \text{span} \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \quad \text{and} \quad E_1 = \text{span} \left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right)$$

Theorem 5.15 The Gram-Schmidt Process

Let $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ be a basis for a subspace W of \mathbb{R}^n and define the following:

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{x}_1, & W_1 &= \text{span}(\mathbf{x}_1) \\ \mathbf{v}_2 &= \mathbf{x}_2 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1, & W_2 &= \text{span}(\mathbf{x}_1, \mathbf{x}_2) \\ \mathbf{v}_3 &= \mathbf{x}_3 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_3}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{x}_3}{\mathbf{v}_2 \cdot \mathbf{v}_2} \right) \mathbf{v}_2, & W_3 &= \text{span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \\ &\vdots & & \\ \mathbf{v}_k &= \mathbf{x}_k - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_k}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{x}_k}{\mathbf{v}_2 \cdot \mathbf{v}_2} \right) \mathbf{v}_2 - \dots \\ &\quad - \left(\frac{\mathbf{v}_{k-1} \cdot \mathbf{x}_k}{\mathbf{v}_{k-1} \cdot \mathbf{v}_{k-1}} \right) \mathbf{v}_{k-1}, & W_k &= \text{span}(\mathbf{x}_1, \dots, \mathbf{x}_k) \end{aligned}$$

Then for each $i = 1, \dots, k$, $\{\mathbf{v}_1, \dots, \mathbf{v}_i\}$ is an orthogonal basis for W_i . In particular, $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is an orthogonal basis for W .

Theorem 5.20 The Spectral Theorem

Let A be an $n \times n$ real matrix. Then A is symmetric if and only if it is orthogonally diagonalizable.

The Spectral Theorem allows us to write a real symmetric matrix A in the form $A = QDQ^T$, where Q is orthogonal and D is diagonal.

The diagonal entries of D are just the eigenvalues of A ,

if the columns of Q are the orthonormal vectors $\mathbf{q}_1, \dots, \mathbf{q}_n$,

spectral decomposition of A .

then, using the column-row representation of the product, we have

$$\begin{aligned}
 A &= QDQ^T = [\mathbf{q}_1 \ \cdots \ \mathbf{q}_n] \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{q}_1^T \\ \vdots \\ \mathbf{q}_n^T \end{bmatrix} \\
 &= [\lambda_1 \mathbf{q}_1 \ \cdots \ \lambda_n \mathbf{q}_n] \begin{bmatrix} \mathbf{q}_1^T \\ \vdots \\ \mathbf{q}_n^T \end{bmatrix} \\
 &= \lambda_1 \mathbf{q}_1 \mathbf{q}_1^T + \lambda_2 \mathbf{q}_2 \mathbf{q}_2^T + \cdots + \lambda_n \mathbf{q}_n \mathbf{q}_n^T
 \end{aligned}$$

spectral decomposition of A .

Each of the terms $\lambda_i \mathbf{q}_i \mathbf{q}_i^T$ is a rank 1 matrix, $\mathbf{q}_i \mathbf{q}_i^T$ is actually the matrix of the projection onto the subspace spanned by \mathbf{q}_i .

projection form of the Spectral Theorem.

25. Let \mathbf{q} be a unit vector in \mathbb{R}^n and let W be the subspace spanned by \mathbf{q} . Show that the orthogonal projection of a vector \mathbf{v} onto W (as defined in Sections 1.2 and 5.2) is given by

$$\text{proj}_W(\mathbf{v}) = (\mathbf{q}\mathbf{q}^T)\mathbf{v}$$

and that the matrix of this projection is thus $\mathbf{q}\mathbf{q}^T$.

[Hint: Remember that, for \mathbf{x} and \mathbf{y} in \mathbb{R}^n , $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y}$.]

$\mathbf{q}\mathbf{q}^T$ 正好就是一個投影矩陣，它專門負責把空間中的任何向量 \mathbf{v} ，直直地投影到單位向量 \mathbf{q} 所代表的那條直線上。

Example 5.19

Find the spectral decomposition of the matrix A from Example 5.18.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Solution From Example 5.18, we have:

$$\lambda_1 = 4, \quad \lambda_2 = 1, \quad \lambda_3 = 1$$
$$\mathbf{q}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \quad \mathbf{q}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad \mathbf{q}_3 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

Example 5.20

Find a 2×2 matrix with eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -2$ and corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

Solution We begin by normalizing the vectors to obtain an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2\}$, with

$$\mathbf{q}_1 = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} \quad \text{and} \quad \mathbf{q}_2 = \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$$