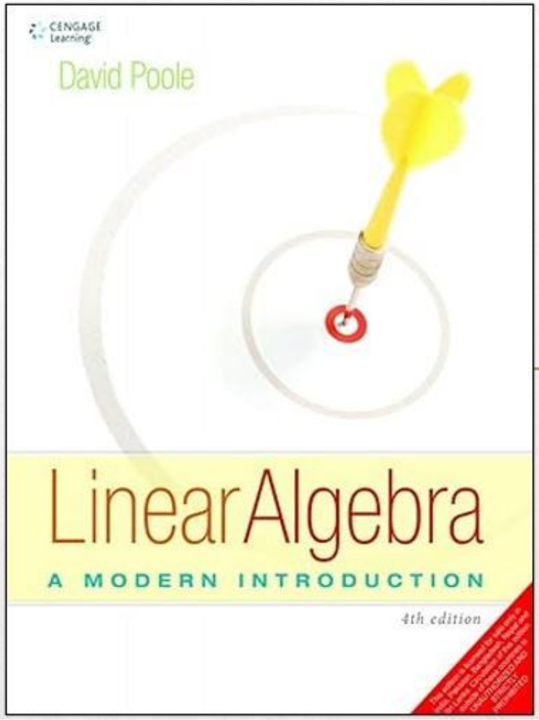

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線性代數 (二)

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5.3 The Gram-Schmidt Process and the *QR* Factorization

https://hmwu.idv.tw

The Gram-Schmidt Process

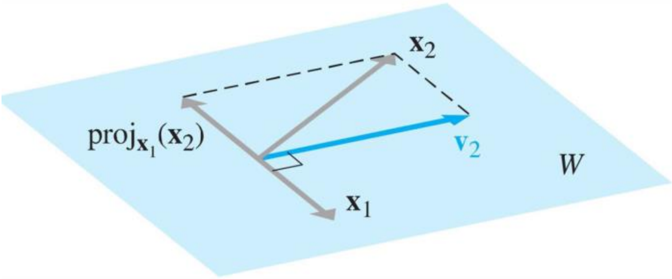
We would like to be able to find an orthogonal basis for a subspace W of $\mathbb{R}^n$.

The idea is to begin with an arbitrary basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ for W and to "orthogonalize" it one vector at a time.

Example 5.12 Let $W = \text{span}(\mathbf{x}_1, \mathbf{x}_2)$, where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Construct an orthogonal basis for W.



Solution Starting with $\mathbf{x}_1$, we get a second vector that is orthogonal to it by taking the component of $\mathbf{x}_2$ orthogonal to $\mathbf{x}_1$ (Figure 5.10).

Then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal set of vectors in W.

Hence, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly independent set and therefore a basis for W, since $\dim W = 2$.

Remark Observe that this method depends on the *order* of the original basis vectors. In Example 5.12, if we had taken $\mathbf{x}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, we would have obtained a different orthogonal basis for W. (Verify this.)

Theorem 5.15 The Gram-Schmidt Process

Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ be a basis for a subspace W of $\mathbb{R}^n$ and define the following:

$$\mathbf{v}_1 = \mathbf{x}_1, \qquad W_1 = \text{span}(\mathbf{x}_1)$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1, \qquad W_2 = \text{span}(\mathbf{x}_1, \mathbf{x}_2)$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_3}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{x}_3}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2, \qquad W_3 = \text{span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$$

$$\vdots$$

$$\mathbf{v}_k = \mathbf{x}_k - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_k}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{x}_k}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 - \cdots$$

$$- \left(\frac{\mathbf{v}_{k-1} \cdot \mathbf{x}_k}{\mathbf{v}_{k-1} \cdot \mathbf{v}_{k-1}}\right)\mathbf{v}_{k-1}, \qquad W_k = \text{span}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$$

Then for each $i = 1, \ldots, k$, $\{\mathbf{v}_1, \ldots, \mathbf{v}_i\}$ is an orthogonal basis for W_i. In particular, $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal basis for W.

Example 5.13 Apply the Gram-Schmidt Process to construct an orthonormal basis for the subspace $W = \text{span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ of $\mathbb{R}^4$, where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

Solution $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a linearly independent set, so it forms a basis for W.

setting $\mathbf{v}_1 = \mathbf{x}_1$. $W_1 = \text{span}(\mathbf{v}_1)$

We now have an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2', \mathbf{v}_3'\}$ for W.

Example 5.13

Apply the Gram-Schmidt Process to construct an orthonormal basis for the subspace $W = \text{span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ of $\mathbb{R}^4$, where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

$$\mathbf{q}_1 = \left(\frac{1}{\|\mathbf{v}_1\|}\right)\mathbf{v}_1 = \left(\frac{1}{2}\right)\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$\mathbf{q}_2 = \left(\frac{1}{\|\mathbf{v}_2'\|}\right)\mathbf{v}_2' = \left(\frac{1}{2\sqrt{5}}\right)\begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2\sqrt{5} \\ 3/2\sqrt{5} \\ 1/2\sqrt{5} \\ 1/2\sqrt{5} \end{bmatrix} = \begin{bmatrix} 3\sqrt{5}/10 \\ 3\sqrt{5}/10 \\ \sqrt{5}/10 \\ \sqrt{5}/10 \end{bmatrix}$$

$$\mathbf{q}_3 = \left(\frac{1}{\|\mathbf{v}_3'\|}\right)\mathbf{v}_3' = \left(\frac{1}{\sqrt{6}}\right)\begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{6} \\ 0 \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} -\sqrt{6}/6 \\ 0 \\ \sqrt{6}/6 \\ \sqrt{6}/3 \end{bmatrix}$$

Then $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ is an orthonormal basis for W.

Example 5.14

Find an orthogonal basis for $\mathbb{R}^3$ that contains the vector

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

now apply the Gram-Schmidt Process to this basis to obtain

$$\mathbf{v}_2 = \mathbf{x}_2 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \left(\tfrac{2}{14}\right)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{7} \\ \frac{5}{7} \\ -\frac{3}{7} \end{bmatrix}, \quad \mathbf{v}_2' = \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_3}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2' \cdot \mathbf{x}_3}{\mathbf{v}_2' \cdot \mathbf{v}_2'}\right)\mathbf{v}_2' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \left(\tfrac{3}{14}\right)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \left(\tfrac{-3}{35}\right)\begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} \frac{-3}{10} \\ 0 \\ \frac{1}{10} \end{bmatrix}, \quad \mathbf{v}_3' = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Then $\{\mathbf{v}_1, \mathbf{v}_2', \mathbf{v}_3'\}$ is an orthogonal basis for $\mathbb{R}^3$ that contains $\mathbf{v}_1$.

Theorem 5.16 The QR Factorization

Let A be an $m \times n$ matrix with linearly independent columns. Then A can be factored as $A = QR$, where Q is an $m \times n$ matrix with orthonormal columns and R is an invertible upper triangular matrix.

A is an $m \times n$ matrix

let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be the (linearly independent) columns of A

let $\mathbf{q}_1, \ldots, \mathbf{q}_n$ be the orthonormal vectors obtained by applying the Gram-Schmidt Process to A with normalizations.

From Theorem 5.15, we know that, for each $i = 1, \ldots, n$,

$$W_i = \operatorname{span}(\mathbf{a}_1, \ldots, \mathbf{a}_i) = \operatorname{span}(\mathbf{q}_1, \ldots, \mathbf{q}_i)$$

Theorem 5.15 The Gram-Schmidt Process

Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ be a basis for a subspace W of $\mathbb{R}^n$ and define the following:

$$\mathbf{v}_1 = \mathbf{x}_1, \qquad W_1 = \operatorname{span}(\mathbf{x}_1)$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1, \qquad W_2 = \operatorname{span}(\mathbf{x}_1, \mathbf{x}_2)$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_3}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{x}_3}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2, \qquad W_3 = \operatorname{span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$$

$$\vdots$$

$$\mathbf{v}_k = \mathbf{x}_k - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_k}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{x}_k}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 - \cdots - \left(\frac{\mathbf{v}_{k-1} \cdot \mathbf{x}_k}{\mathbf{v}_{k-1} \cdot \mathbf{v}_{k-1}}\right)\mathbf{v}_{k-1}, \qquad W_k = \operatorname{span}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$$

Then for each $i = 1, \ldots, k$, $\{\mathbf{v}_1, \ldots, \mathbf{v}_i\}$ is an orthogonal basis for W_i. In particular, $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal basis for W.

Example 5.15

Find a QR factorization of

$$A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Solution The columns of A are just the vectors from Example 5.13. The orthonormal basis for col(A) produced by the Gram-Schmidt Process was

$$Q = [\mathbf{q}_1 \quad \mathbf{q}_2 \quad \mathbf{q}_3] = \begin{bmatrix} 1/2 & 3\sqrt{5}/10 & -\sqrt{6}/6 \\ -1/2 & 3\sqrt{5}/10 & 0 \\ -1/2 & \sqrt{5}/10 & \sqrt{6}/6 \\ 1/2 & \sqrt{5}/10 & \sqrt{6}/3 \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ 3\sqrt{5}/10 & 3\sqrt{5}/10 & \sqrt{5}/10 & \sqrt{5}/10 \\ -\sqrt{6}/6 & 0 & \sqrt{6}/6 & \sqrt{6}/3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1/2 \\ 0 & \sqrt{5} & 3\sqrt{5}/2 \\ 0 & 0 & \sqrt{6}/2 \end{bmatrix}$$

Exercises 5.3

2, 5, 7, 11, 16, 17