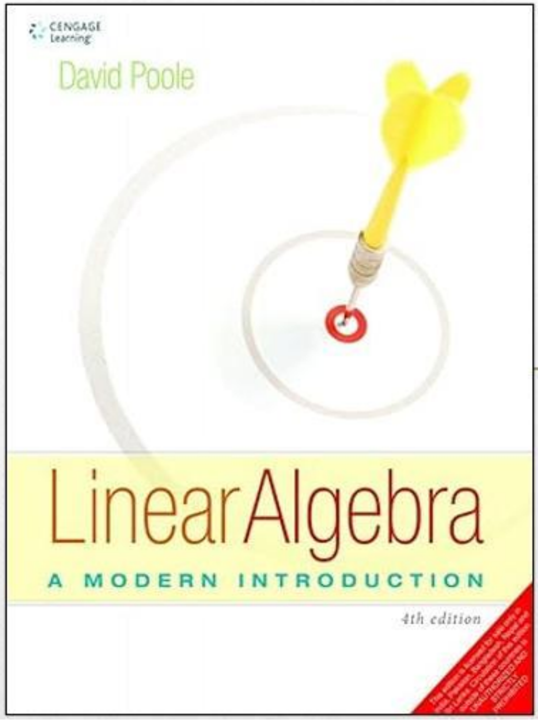

114-2
線性代數 (二)

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5.2
Orthogonal Complements and Orthogonal Projections

https://hmwu.idv.tw

Definition Let W be a subspace of $\mathbb{R}^n$. We say that a vector $\mathbf{v}$ in $\mathbb{R}^n$ is ***orthogonal to*** W if $\mathbf{v}$ is orthogonal to every vector in W. The set of all vectors that are orthogonal to W is called the ***orthogonal complement of*** W, denoted $W^\perp$. That is,

$$W^\perp = \{\mathbf{v} \text{ in } \mathbb{R}^n : \mathbf{v} \cdot \mathbf{w} = 0 \quad \text{for all } \mathbf{w} \text{ in } W\}$$

Example 5.8 If W is a plane through the origin in $\mathbb{R}^3$ and ℓ is the line through the origin perpendicular to W (i.e., parallel to the normal vector to W), then every vector $\mathbf{v}$ on ℓ is orthogonal to every vector $\mathbf{w}$ in W; hence, $\ell = W^\perp$. Moreover, W consists *precisely* of those vectors $\mathbf{w}$ that are orthogonal to every $\mathbf{v}$ on ℓ; hence, we also have $W = \ell^\perp$.

$W^\perp$ is pronounced "W perp."

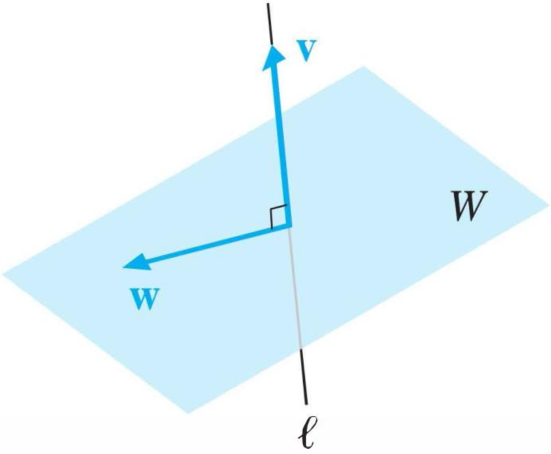
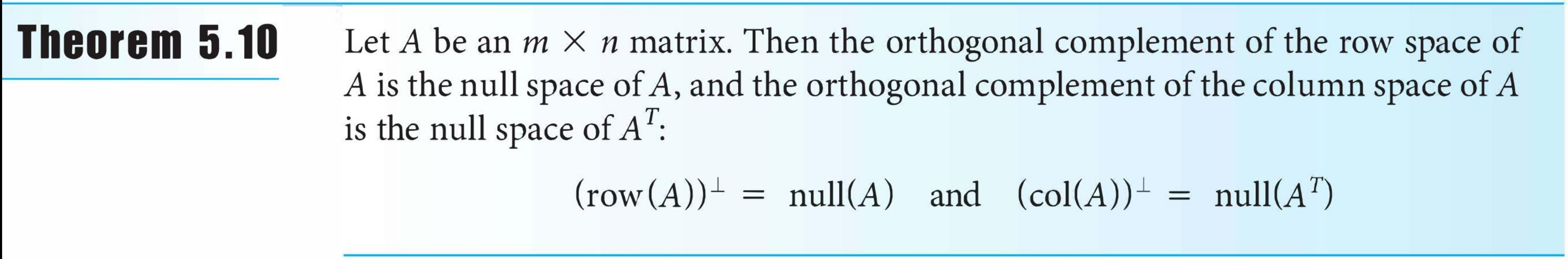


Figure 5.5
$\ell = W^\perp$ and $W = \ell^\perp$

Theorem 5.9 Let W be a subspace of $\mathbb{R}^n$.

a. $W^\perp$ is a subspace of $\mathbb{R}^n$.
b. $(W^\perp)^\perp = W$
c. $W \cap W^\perp = \{\mathbf{0}\}$
d. If $W = \text{span}(\mathbf{w}_1, \ldots, \mathbf{w}_k)$, then $\mathbf{v}$ is in $W^\perp$ if and only if $\mathbf{v} \cdot \mathbf{w}_i = 0$ for all $i = 1, \ldots, k$.

Proof (a) Since $\mathbf{0} \cdot \mathbf{w} = 0$ for all $\mathbf{w}$ in W, $\mathbf{0}$ is in $W^\perp$.

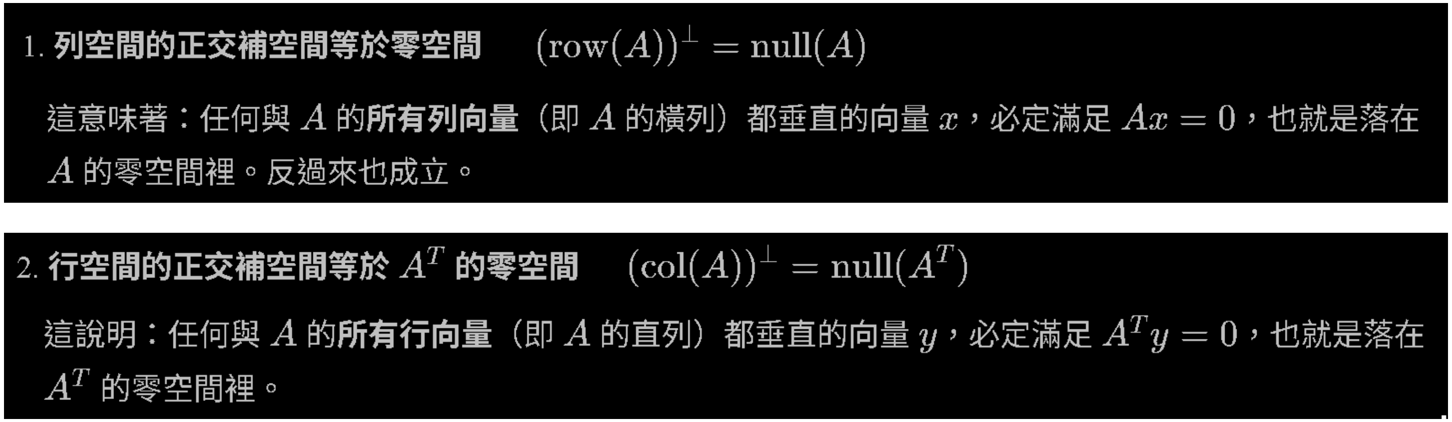


Theorem 5.10 Let A be an $m \times n$ matrix. Then the orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T:

$$(\text{row}(A))^{\perp} = \text{null}(A) \quad \text{and} \quad (\text{col}(A))^{\perp} = \text{null}(A^T)$$

To prove the second identity, we simply replace A by A^T and use the fact that $\text{row}(A^T) = \text{col}(A)$.

線性代數中四個基本子空間之間的正交關係。



1. **列空間的正交補空間等於零空間** $(\text{row}(A))^{\perp} = \text{null}(A)$

這意味著：任何與 A 的**所有列向量**（即 A 的橫列）都垂直的向量 x，必定滿足 $Ax = 0$，也就是落在 A 的零空間裡。反過來也成立。

2. **行空間的正交補空間等於 A^T 的零空間** $(\text{col}(A))^{\perp} = \text{null}(A^T)$

這說明：任何與 A 的**所有行向量**（即 A 的直列）都垂直的向量 y，必定滿足 $A^Ty = 0$，也就是落在 A^T 的零空間裡。

These four subspaces are called the ***fundamental subspaces*** of the $m \times n$ matrix A.

Thus, an $m \times n$ matrix has four subspaces: $\text{row}(A)$, $\text{null}(A)$, $\text{col}(A)$, and $\text{null}(A^T)$.

The first two are orthogonal complements in $\mathbb{R}^n$,

the last two are orthogonal complements in $\mathbb{R}^m$.

- 列空間 $\text{row}(A)$ 與零空間 $\text{null}(A)$ 互為正交補空間，且在 $\mathbb{R}^n$ 中將空間分割成兩個互補且正交的子空間。
- 行空間 $\text{col}(A)$ 與左零空間 $\text{null}(A^T)$ 互為正交補空間，在 $\mathbb{R}^m$ 中亦然。

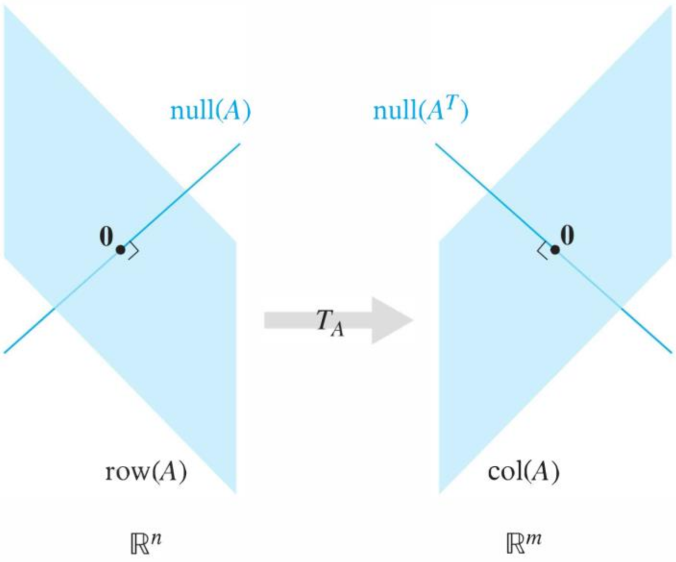


Figure 5.6
The four fundamental subspaces

Example 5.9

Find bases for the four fundamental subspaces of

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix}$$

Solution In Examples 3.45, 3.47, and 3.48, we computed bases for the row space, column space, and null space of A.

We found that $\text{row}(A) = \text{span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, where

$$\mathbf{u}_1 = [1 \;\; 0 \;\; 1 \;\; 0 \;\; -1], \quad \mathbf{u}_2 = [0 \;\; 1 \;\; 2 \;\; 0 \;\; 3], \quad \mathbf{u}_3 = [0 \;\; 0 \;\; 0 \;\; 1 \;\; 4]$$

Also, $\text{null}(A) = \text{span}(\mathbf{x}_1, \mathbf{x}_2)$, where

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ -3 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

The column space of A is $\text{col}(A) = \text{span}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$,

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

and it is easy to check that this vector is orthogonal to $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

Example 3.45 Find a basis for the row space of

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix}$$

Solution The reduced row echelon form of A is

$$R = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By Theorem 3.20, $\text{row}(A) = \text{row}(R)$, so it is enough to find a basis for the row space of R. But $\text{row}(R)$ is clearly spanned by its nonzero rows, and it is easy to check that the staircase pattern forces the first three rows of R to be linearly independent. (This is a general fact, one that you will need to establish to prove Exercise 33.) Therefore, a basis for the row space of A is

$$\{[1 \quad 0 \quad 1 \quad 0 \quad -1], [0 \quad 1 \quad 2 \quad 0 \quad 3], [0 \quad 0 \quad 0 \quad 1 \quad 4]\}$$

Example 3.47 Find a basis for the column space of the matrix from Example 3.45,

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix}$$

Solution Let $\mathbf{a}_i$ denote a column vector of A and let $\mathbf{r}_i$ denote a column vector of the reduced echelon form

$$R = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can quickly see by inspection that $\mathbf{r}_3 = \mathbf{r}_1 + 2\mathbf{r}_2$ and $\mathbf{r}_5 = -\mathbf{r}_1 + 3\mathbf{r}_2 + 4\mathbf{r}_4$.

The remaining column vectors, $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_4$, are linearly independent,

Use the columns of A that correspond to the

columns of R *containing the leading 1s*. A basis for $\text{col}(A)$ is

$$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

Example 3.48 Find a basis for the null space of matrix A from Example 3.47.

Solution

the solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$.

$$[R \mid \mathbf{0}] = \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

We get $x_1 = -x_3 + x_5$, $x_2 = -2x_3 - 3x_5$, and $x_4 = -4x_5$.

Setting $x_3 = s$ and $x_5 = t$, we obtain

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s + t \\ -2s - 3t \\ s \\ -4t \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -3 \\ 0 \\ -4 \\ 1 \end{bmatrix} = s\mathbf{u} + t\mathbf{v}$$

Thus, $\mathbf{u}$ and $\mathbf{v}$ span $\text{null}(A)$, and since they are linearly independent, they form a basis for $\text{null}(A)$.

Orthogonal Projections

Recall that, in $\mathbb{R}^2$, the projection of a vector $\mathbf{v}$ onto a nonzero vector $\mathbf{u}$ is given by

$$\operatorname{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}$$

Furthermore, the vector $\operatorname{perp}_{\mathbf{u}}(\mathbf{v}) = \mathbf{v} - \operatorname{proj}_{\mathbf{u}}(\mathbf{v})$ is orthogonal to $\operatorname{proj}_{\mathbf{u}}(\mathbf{v})$, and we can decompose $\mathbf{v}$ as

$$\mathbf{v} = \operatorname{proj}_{\mathbf{u}}(\mathbf{v}) + \operatorname{perp}_{\mathbf{u}}(\mathbf{v})$$

as shown in Figure 5.7.

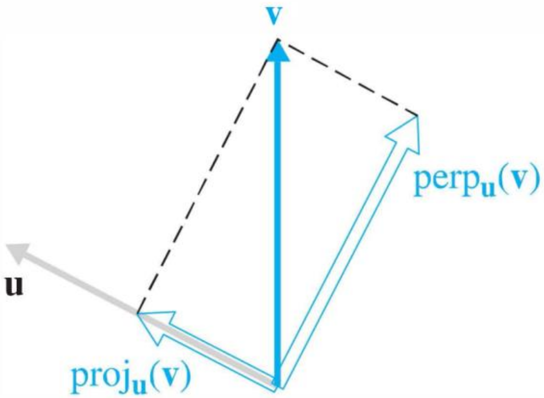


Figure 5.7

$\mathbf{v} = \operatorname{proj}_{\mathbf{u}}(\mathbf{v}) + \operatorname{perp}_{\mathbf{u}}(\mathbf{v})$

If we let $W = \operatorname{span}(\mathbf{u})$, then $\mathbf{w} = \operatorname{proj}_{\mathbf{u}}(\mathbf{v})$ is in W and $\mathbf{w}^{\perp} = \operatorname{perp}_{\mathbf{u}}(\mathbf{v})$ is in $W^{\perp}$.

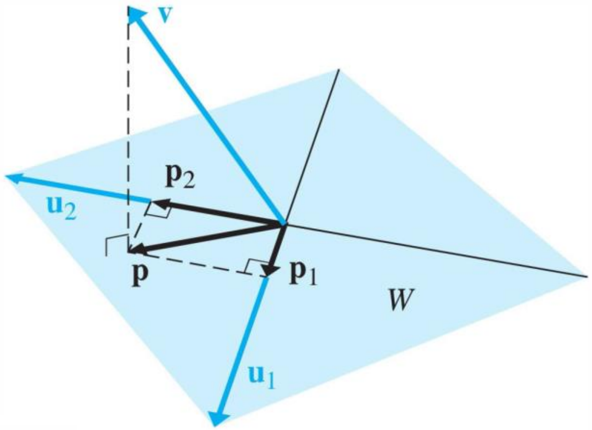
Definition Let W be a subspace of $\mathbb{R}^n$ and let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be an orthogonal basis for W. For any vector $\mathbf{v}$ in $\mathbb{R}^n$, the ***orthogonal projection of* v *onto* W** is defined as

$$\operatorname{proj}_W(\mathbf{v}) = \left(\frac{\mathbf{u}_1 \cdot \mathbf{v}}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1 + \cdots + \left(\frac{\mathbf{u}_k \cdot \mathbf{v}}{\mathbf{u}_k \cdot \mathbf{u}_k}\right)\mathbf{u}_k$$

The ***component of* v *orthogonal to* W** is the vector

$$\operatorname{perp}_W(\mathbf{v}) = \mathbf{v} - \operatorname{proj}_W(\mathbf{v})$$

$$\operatorname{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\right)\mathbf{u}$$



$$\operatorname{proj}_W(\mathbf{v}) = \operatorname{proj}_{\mathbf{u}_1}(\mathbf{v}) + \cdots + \operatorname{proj}_{\mathbf{u}_k}(\mathbf{v})$$

if $\mathbf{w}$ is in the subspace W of $\mathbb{R}^n$, which has orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, then

$$\begin{aligned}\mathbf{w} &= \left(\frac{\mathbf{w} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 + \cdots + \left(\frac{\mathbf{w} \cdot \mathbf{v}_k}{\mathbf{v}_k \cdot \mathbf{v}_k}\right)\mathbf{v}_k \\ &= \operatorname{proj}_{\mathbf{v}_1}(\mathbf{w}) + \cdots + \operatorname{proj}_{\mathbf{v}_k}(\mathbf{w})\end{aligned}$$

Thus, $\mathbf{w}$ is decomposed into a sum of orthogonal projections onto mutually orthogonal one-dimensional subspaces of W.

Example 5.11

Let W be the plane in $\mathbb{R}^3$ with equation $x - y + 2z = 0$, and let $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$. Find the orthogonal projection of $\mathbf{v}$ onto W and the component of $\mathbf{v}$ orthogonal to W.

Solution In Example 5.3, we found an orthogonal basis for W. Taking

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

we have

$$\begin{array}{ll} \mathbf{u}_1 \cdot \mathbf{v} = 2 & \mathbf{u}_2 \cdot \mathbf{v} = -2 \\ \mathbf{u}_1 \cdot \mathbf{u}_1 = 2 & \mathbf{u}_2 \cdot \mathbf{u}_2 = 3 \end{array}$$

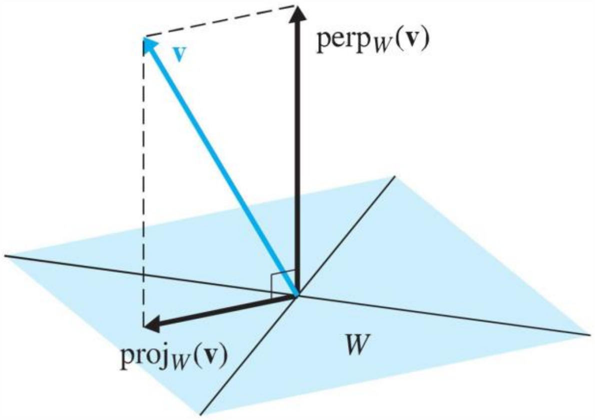


Figure 5.9

$\mathbf{v} = \text{proj}_W(\mathbf{v}) + \text{perp}_W(\mathbf{v})$

Theorem 5.11 **The Orthogonal Decomposition Theorem**

Let W be a subspace of $\mathbb{R}^n$ and let $\mathbf{v}$ be a vector in $\mathbb{R}^n$. Then there are unique vectors $\mathbf{w}$ in W and $\mathbf{w}^\perp$ in $W^\perp$ such that

$$\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$$

Example 5.11 illustrated the Orthogonal Decomposition Theorem.

When W is the subspace of $\mathbb{R}^3$ given by the plane with equation $x - y + 2z = 0$,

the orthogonal decomposition of

$$\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \text{ with respect to } W \text{ is } \mathbf{v} = \mathbf{w} + \mathbf{w}^\perp\text{, where}$$

$$\mathbf{w} = \operatorname{proj}_W(\mathbf{v}) = \begin{bmatrix} \frac{5}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} \quad \text{and} \quad \mathbf{w}^\perp = \operatorname{perp}_W(\mathbf{v}) = \begin{bmatrix} \frac{4}{3} \\ -\frac{4}{3} \\ \frac{8}{3} \end{bmatrix}$$

Corollary 5.12 If W is a subspace of $\mathbb{R}^n$, then

$$(W^\perp)^\perp = W$$

Theorem 5.13 If W is a subspace of $\mathbb{R}^n$, then

$$\dim W + \dim W^\perp = n$$

Corollary 5.14 **The Rank Theorem**

If A is an $m \times n$ matrix, then $\operatorname{rank}(A) + \operatorname{nullity}(A) = n$

Exercises 5.2

2, 5, 7, 9, 12, 15, 20