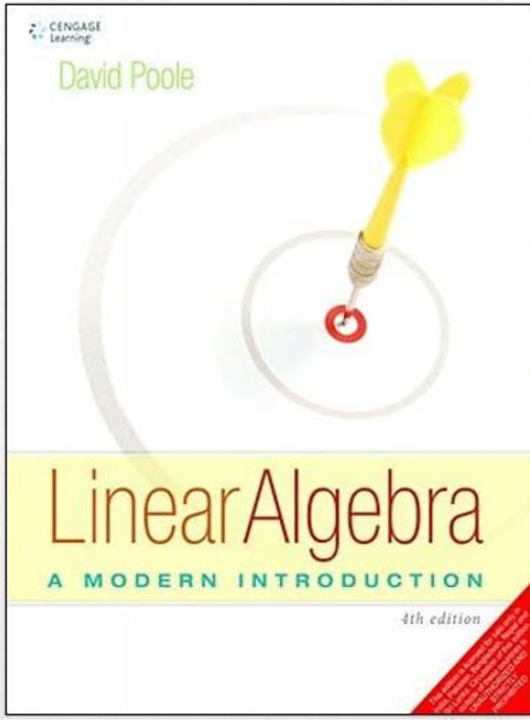

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線性代數 (二)

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4.4 Similarity and Diagonalization

https://hmwu.idv.tw

Similar Matrices

Definition Let A and B be $n \times n$ matrices. We say that A ***is similar to*** B if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = B$. If A is similar to B, we write $A \sim B$.

Remarks

- If $A \sim B$, we can write, equivalently, that $A = PBP^{-1}$ or $AP = PB$.

Example 4.22 Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$. Then $A \sim B$,

Theorem 4.21 Let A, B, and C be $n \times n$ matrices.

a. $A \sim A$
b. If $A \sim B$, then $B \sim A$.
c. If $A \sim B$ and $B \sim C$, then $A \sim C$.

Remark Any relation satisfying the three properties of Theorem 4.21 is called an ***equivalence relation.***

Theorem 4.22 Let A and B be $n \times n$ matrices with $A \sim B$. Then

a. $\det A = \det B$
b. A is invertible if and only if B is invertible.
c. A and B have the same rank.
d. A and B have the same characteristic polynomial.
e. A and B have the same eigenvalues.
f. $A^m \sim B^m$ for all integers $m \geq 0$.
g. If A is invertible, then $A^m \sim B^m$ for all integers m.

Proof

If $A \sim B$, then $P^{-1}AP = B$ for some invertible matrix P.

Example 4.23

(a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ are not similar, since $\det A = -3$ but $\det B = 3$.

(b) $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$ are not similar, since the characteristic polynomial of A is $\lambda^2 - 3\lambda - 4$ while that of B is $\lambda^2 - 4$. (Check this.) Note that A and B do have the same determinant and rank, however.

Diagonalization

Definition An $n \times n$ matrix A is ***diagonalizable*** if there is a diagonal matrix D such that A is similar to D—that is, if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = D$.

Example 4.24

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \text{ is diagonalizable}$$

$$\text{if } P = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix} \text{ and } D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}, \text{ then } P^{-1}AP = D$$

equivalent statement $AP = PD$

Theorem 4.23 Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if A has n linearly independent eigenvectors.

More precisely, there exist an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ if and only if the columns of P are n linearly independent eigenvectors of A and the diagonal entries of D are the eigenvalues of A corresponding to the eigenvectors in P in the same order.

Proof Suppose $A \sim D$ (diagonal matrix) via $P^{-1}AP = D$, $AP = PD$.

Let the columns of P be $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$

let the diagonal entries of D be $\lambda_1, \lambda_2, \ldots, \lambda_n.$

Then $A[\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_n] = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_n]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$

or $[A\mathbf{p}_1 \quad A\mathbf{p}_2 \quad \cdots \quad A\mathbf{p}_n] = [\lambda_1\mathbf{p}_1 \quad \lambda_2\mathbf{p}_2 \quad \cdots \quad \lambda_n\mathbf{p}_n]$

we have $A\mathbf{p}_1 = \lambda_1\mathbf{p}_1, A\mathbf{p}_2 = \lambda_2\mathbf{p}_2, \ldots, A\mathbf{p}_n = \lambda_n\mathbf{p}_n$

which proves that the column vectors of P are eigenvectors of A whose corresponding eigenvalues are the diagonal entries of D in the same order.

Since P is invertible, its columns are linearly independent,

by the Fundamental Theorem of Invertible Matrices.

Theorem 4.23 Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if A has n linearly independent eigenvectors.

More precisely, there exist an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ if and only if the columns of P are n linearly independent eigenvectors of A and the diagonal entries of D are the eigenvalues of A corresponding to the eigenvectors in P in the same order.

Proof Conversely, if A has n linearly independent eigenvectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively,

then $A\mathbf{p}_1 = \lambda_1\mathbf{p}_1, A\mathbf{p}_2 = \lambda_2\mathbf{p}_2, \ldots, A\mathbf{p}_n = \lambda_n\mathbf{p}_n$

This implies Equation (2) above, which is equivalent to Equation (1).

$$[A\mathbf{p}_1 \quad A\mathbf{p}_2 \quad \cdots \quad A\mathbf{p}_n] = [\lambda_1\mathbf{p}_1 \quad \lambda_2\mathbf{p}_2 \quad \cdots \quad \lambda_n\mathbf{p}_n] \qquad (2)$$

Consequently,

if we take P to be the $n \times n$ matrix with columns $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$, then Equation (1) becomes $AP = PD$.

Since the columns of P are linearly independent, the Fundamental Theorem of Invertible Matrices implies that P is invertible, so $P^{-1}AP = D$;

that is, A is diagonalizable.

Example 4.25 If possible, find a matrix P that diagonalizes

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$$

Solution in Example 4.18

eigenvalues $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$.

E_1 has basis $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. E_2 has basis $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$.

Theorem 4.23 Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if A has n linearly independent eigenvectors.

More precisely, there exist an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ if and only if the columns of P are n linearly independent eigenvectors of A and the diagonal entries of D are the eigenvalues of A corresponding to the eigenvectors in P in the same order.

Example 4.26 If possible, find a matrix P that diagonalizes

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$$

Solution Example 4.19.

eigenvalues: $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = -2$,

eigenspaces:

$$E_0 \text{ has basis } \mathbf{p}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{p}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

$$E_{-2} \text{ has basis } \mathbf{p}_3 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}.$$

Theorem 4.24 Let A be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of A. If $\mathcal{B}_i$ is a basis for the eigenspace E_{λ_i}, then $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k$ (i.e., the total collection of basis vectors for all of the eigenspaces) is linearly independent.

Theorem 4.25 If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable.

Example 4.27

The matrix

$$A = \begin{bmatrix} 2 & -3 & 7 \\ 0 & 5 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

has eigenvalues $\lambda_1 = 2$, $\lambda_2 = 5$, and $\lambda_3 = -1$.

Since these are three distinct eigenvalues for a 3×3 matrix,

A is diagonalizable, by Theorem 4.25.

Lemma 4.26 If A is an $n \times n$ matrix, then the geometric multiplicity of each eigenvalue is less than or equal to its algebraic multiplicity.

重要關係是：

$$1 \le \text{幾何重數} \le \text{代數重數}$$

直觀上：

代數重數：這個特徵值在「方程式裡」重複幾次。
幾何重數：這個特徵值實際上能提供幾個獨立方向的特徵向量。

代數重數看「根重複幾次」；
幾何重數看「有幾個獨立的特徵向量方向」。

Theorem 4.27 **The Diagonalization Theorem**

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_k$. The following statements are equivalent:

a. A is diagonalizable.
b. The union $\mathcal{B}$ of the bases of the eigenspaces of A (as in Theorem 4.24) contains n vectors.
c. The algebraic multiplicity of each eigenvalue equals its geometric multiplicity.

Example 4.28

(a) The matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$ from Example 4.18 has two distinct eigenvalues,

$\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$.

$$E_1 = \left\{ \begin{bmatrix} t \\ t \\ t \end{bmatrix} \right\} = \left\{ t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

Since the eigenvalue $\lambda_1 = \lambda_2 = 1$ has algebraic multiplicity 2 but geometric multiplicity 1, A is not diagonalizable, by the Diagonalization Theorem.

$$E_2 = \left\{ \begin{bmatrix} \frac{1}{4}t \\ \frac{1}{2}t \\ t \end{bmatrix} \right\} = \left\{ t \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\} = \text{span}\left(\begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right)$$

(b) The matrix $A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$ from Example 4.19 also has two distinct eigenvalues, $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = -2$.

The eigenvalue 0 has algebraic and geometric multiplicity 2,

the eigenvalue -2 has algebraic and geometric multiplicity 1.

Thus, this matrix *is* diagonalizable, by the Diagonalization Theorem.

$$E_0 = \left\{ \begin{bmatrix} t \\ s \\ t \end{bmatrix} \right\} = \left\{ s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$E_{-2} = \left\{ \begin{bmatrix} -t \\ 3t \\ t \end{bmatrix} \right\} = \left\{ t \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right\} = \text{span}\left(\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right)$$

Example 4.29 Compute A^{10} if $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$.

Solution In Example 4.21, eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 2$, with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Exercises 4.4 2, 3, 6, 9, 12, 18, 21, 28, 39, 46