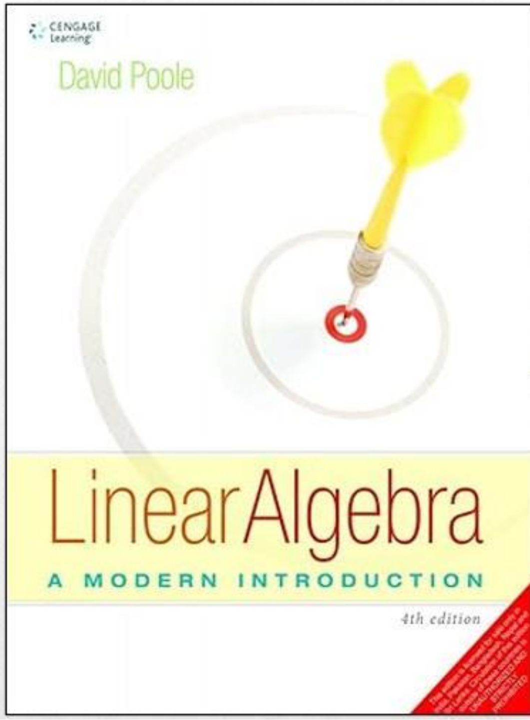

114-2

線性代數 (二)

國立政治大學
統計學系
吳漢銘

4.3 Eigenvalues and Eigenvectors of $n \times n$ Matrices

https://hmwu.idv.tw

The eigenvalues of a square matrix A are precisely the solutions λ of the equation

$$\det(A - \lambda I) = 0$$

The equation $\det(A - \lambda I) = 0$ is called the ***characteristic equation*** of A.

the procedure we will follow (for now) to find the eigenvalues and eigenvectors (eigenspaces) of a matrix.

Let A be an $n \times n$ matrix.

1. Compute the characteristic polynomial $\det(A - \lambda I)$ of A.
2. Find the eigenvalues of A by solving the characteristic equation $\det(A - \lambda I) = 0$ for λ.
3. For each eigenvalue λ, find the null space of the matrix $A - \lambda I$. This is the eigenspace E_λ, the nonzero vectors of which are the eigenvectors of A corresponding to λ.
4. Find a basis for each eigenspace.

Example 4.18 Find the eigenvalues and the corresponding eigenspaces of

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$$

Solution The characteristic polynomial is

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4 - \lambda \end{vmatrix}$$

代數重數

在線性代數裡，`algebraic multiplicity` 指的是某個特徵值作為特徵多項式根時的重複次數。
例如若特徵多項式是 $(\lambda - 2)^3(\lambda + 1)$
那麼特徵值 2 的 algebraic multiplicity 是 3，特徵值 -1 的 algebraic multiplicity 是 1。

Example 4.19 Find the eigenvalues and the corresponding eigenspaces of

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$$

For $\lambda_1 = \lambda_2 = 0$, we compute

$$E_0 = \left\{ \begin{bmatrix} t \\ s \\ t \end{bmatrix} \right\} = \left\{ s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

It follows that $\lambda_1 = \lambda_2 = 0$ has geometric multiplicity 2 and $\lambda_3 = -2$ has geometric multiplicity 1. (Note that the algebraic multiplicity equals the geometric multiplicity for each eigenvalue.)

幾何重數：這個特徵值實際上能提供幾個獨立方向的特徵向量。

Theorem 4.15 The eigenvalues of a triangular matrix are the entries on its main diagonal.

Example 4.20 The eigenvalues of

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & 0 & 3 & 0 \\ 5 & 7 & 4 & -2 \end{bmatrix}$$

are $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 3$, and $\lambda_4 = -2$, by Theorem 4.15. Indeed, the characteristic polynomial is just $(2 - \lambda)(1 - \lambda)(3 - \lambda)(-2 - \lambda)$.

Eigenvalues capture much important information about the behavior of a matrix. Once we know the eigenvalues of a matrix, we can deduce a great many things without doing any more work. The next theorem is one of the most important in this regard.

Theorem 4.16 A square matrix A is invertible if and only if 0 is *not* an eigenvalue of A.

Theorem 4.17

The Fundamental Theorem of Invertible Matrices: Version 3

Let A be an $n \times n$ matrix. The following statements are equivalent:

a. A is invertible.
b. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.
c. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
d. The reduced row echelon form of A is I_n.
e. A is a product of elementary matrices.
f. $\operatorname{rank}(A) = n$
g. $\operatorname{nullity}(A) = 0$
h. The column vectors of A are linearly independent.
i. The column vectors of A span $\mathbb{R}^n$.
j. The column vectors of A form a basis for $\mathbb{R}^n$.
k. The row vectors of A are linearly independent.
l. The row vectors of A span $\mathbb{R}^n$.
m. The row vectors of A form a basis for $\mathbb{R}^n$.
n. $\det A \neq 0$
o. 0 is not an eigenvalue of A.

rank「秩」表示矩陣 A 中有多少個彼此線性獨立的欄向量（也等價於列向量）。
換句話說，rank 告訴你這個矩陣真正提供了多少個「獨立資訊」。

- nullity「零度」或「核度」。它是解齊次方程$Ax = 0$時，自由變數的個數，也就是零空間（null space）的維度。
- $nullity(A) = k$ 表示有 k 個自由方向可以讓 $Ax = 0$ 成立。
- 直觀上：nullity 越大，表示 $Ax = 0$ 有越多非零解。nullity = 0，表示只有零解 $x = 0$

rank-nullity theorem：

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n$$

A 的欄向量線性獨立

意思是若 $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n = 0$

只有 $c_1 = c_2 = \cdots = c_n = 0$ 才成立

那麼這些欄向量就是線性獨立。

Theorem 4.18

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Let A be a square matrix with eigenvalue λ and corresponding eigenvector $\mathbf{x}$.

a. For any positive integer n, λ^n is an eigenvalue of A^n with corresponding eigenvector $\mathbf{x}$.
b. If A is invertible, then $1/\lambda$ is an eigenvalue of A^{-1} with corresponding eigenvector $\mathbf{x}$.
c. If A is invertible, then for any integer n, λ^n is an eigenvalue of A^n with corresponding eigenvector $\mathbf{x}$.

Example 4.21

Compute $\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^{10} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

Solution Let $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$;

Theorem 4.19

Suppose the $n \times n$ matrix A has eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$. If $\mathbf{x}$ is a vector in $\mathbb{R}^n$ that can be expressed as a linear combination of these eigenvectors—say,

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m$$

then, for any integer k,

$$A^k\mathbf{x} = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + \cdots + c_m\lambda_m^k\mathbf{v}_m$$

Theorem 4.20

Let A be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be distinct eigenvalues of A with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$. Then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly independent.

Exercises 4.3

1, 6, 15, 17, 23