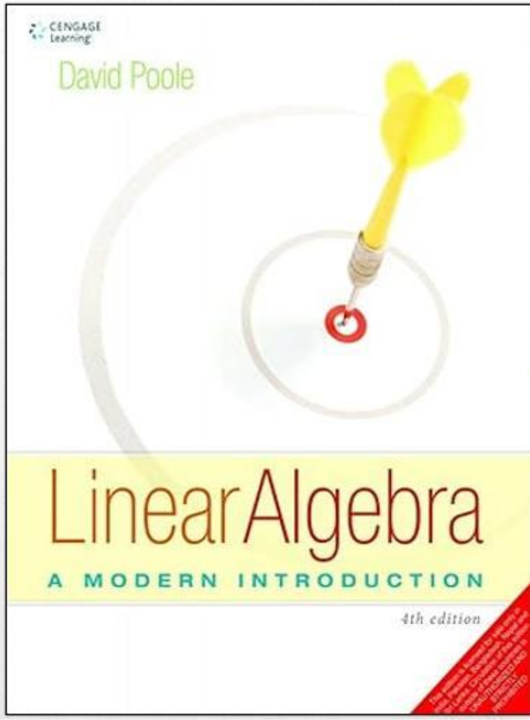

114-2

線性代數 (二)

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4.2 Determinants

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Recall that the determinant of the 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is

$$\det A = a_{11}a_{22} - a_{12}a_{21}$$

幾何意義（最重要的觀點）

行列式在幾何上的核心意義是：**它表示線性轉換對「體積（或面積）」的放大或縮小比例**。

設 $A \in \mathbb{R}^{n\times n}$，把它看成一個線性轉換

$$x \mapsto Ax.$$

當這個轉換作用在空間中的圖形時，它會改變圖形的形狀與大小。
行列式的絕對值 $|\det(A)|$ 就是這個轉換對「n 維體積」的縮放倍率。

二維情況（最直觀）

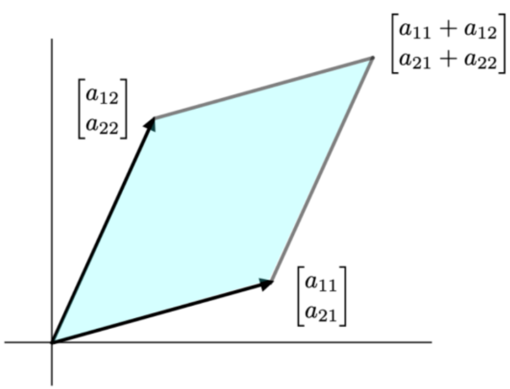
在二維空間中，可以把單位正方形（面積 = 1）經過矩陣 A 變換。
變換後會得到一個平行四邊形。
這個平行四邊形的面積正好是 $|\det(A)|.$

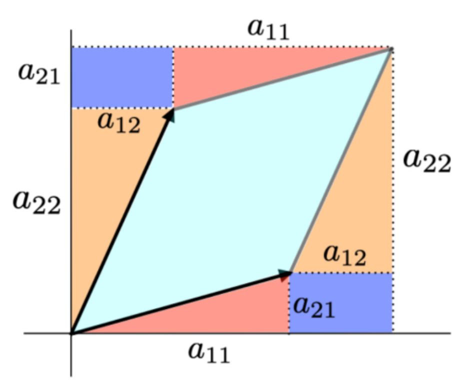
如果：

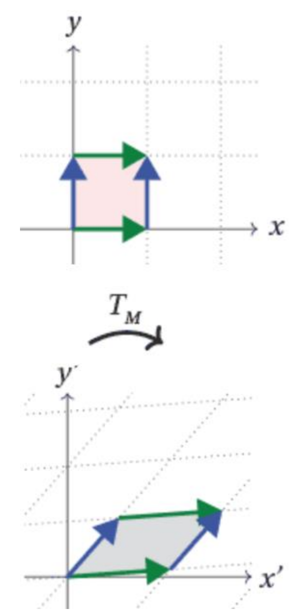
- $|\det(A)| > 1$：面積被放大
- $0 < |\det(A)| < 1$：面積被縮小
- $\det(A) = 0$：面積變成 0（整個圖形被壓扁成一條線）

The determinant

Let's compute the area of the parallelogram spanned by the columns of a matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$







$$\begin{aligned} \text{area of parallelogram} &= (a_{11}+a_{12})(a_{21}+a_{22}) - a_{12}a_{22} - a_{11}a_{21} - 2a_{21}a_{12} \\ &= a_{11}a_{21} + a_{11}a_{22} + a_{12}a_{21} + a_{12}a_{22} - a_{12}a_{22} - a_{11}a_{21} - 2a_{21}a_{12} \\ &= a_{11}a_{22} - a_{21}a_{12} \end{aligned}$$

This number $a_{11}a_{22} - a_{21}a_{12}$ is called the "determinant" of A

The determinant of a matrix A is sometimes also denoted by $|A|$, so for the 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ we may also write

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

inverse of a matrix.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Definition Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then the ***determinant*** of A is the scalar

$$\det A = |A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \quad (1)$$

Equation (1). If we denote by A_{ij} the submatrix of a matrix A obtained by deleting row i and column j, then we may abbreviate Equation (1) as

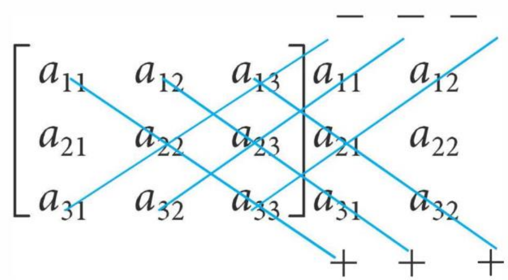
$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13} \\ &= \sum_{j=1}^{3} (-1)^{1+j} a_{1j} \det A_{1j} \end{aligned}$$

For any square matrix A, $\det A_{ij}$ is called the ***(i, j)-minor*** of A.

Example 4.8

Compute the determinant of

$$A = \begin{bmatrix} 5 & -3 & 2 \\ 1 & 0 & 2 \\ 2 & -1 & 3 \end{bmatrix}$$



(2)

This method gives

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

Example 4.9 Calculate the determinant of the matrix in Example 4.8 using the method shown in (2).

Definition Let $A = [a_{ij}]$ be an $n \times n$ matrix, where $n \geq 2$. Then the ***determinant*** of A is the scalar

$$\det A = |A| = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j} \qquad (3)$$

cofactor expansion along the first row.

It is convenient to combine a minor with its plus or minus sign. To this end, we define the ***(i, j)-cofactor of A*** to be

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

With this notation, definition (3) becomes

$$\det A = \sum_{j=1}^{n} a_{1j} C_{1j} \qquad (4)$$

Theorem 4.1 **The Laplace Expansion Theorem**

The determinant of an $n \times n$ matrix $A = [a_{ij}]$, where $n \geq 2$, can be computed as

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij} \tag{5}$$

(which is the ***cofactor expansion along the ith row***) and also as

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}C_{ij} \tag{6}$$

(the ***cofactor expansion along the jth column***).

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

"checkerboard" pattern:

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Example 4.10

Compute the determinant of the matrix

$$A = \begin{bmatrix} 5 & -3 & 2 \\ 1 & 0 & 2 \\ 2 & -1 & 3 \end{bmatrix}$$

by (a) cofactor expansion along the third row and (b) cofactor expansion along the second column.

Example 4.11

Compute the determinant of

$$A = \begin{bmatrix} 2 & -3 & 0 & 1 \\ 5 & 4 & 2 & 0 \\ 1 & -1 & 0 & 3 \\ -2 & 1 & 0 & 0 \end{bmatrix}$$

Example 4.12

Compute the determinant of

$$A = \begin{bmatrix} 2 & -3 & 1 & 0 & 4 \\ 0 & 3 & 2 & 5 & 7 \\ 0 & 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Theorem 4.2

The determinant of a triangular matrix is the product of the entries on its main diagonal. Specifically, if $A = [a_{ij}]$ is an $n \times n$ triangular matrix, then

$$\det A = a_{11}a_{22}\cdots a_{nn}$$

Theorem 4.3 Let $A = [a_{ij}]$ be a square matrix.

a. If A has a zero row (column), then $\det A = 0$.
b. If B is obtained by interchanging two rows (columns) of A, then $\det B = -\det A$.
c. If A has two identical rows (columns), then $\det A = 0$.
d. If B is obtained by multiplying a row (column) of A by k, then $\det B = k \det A$.
e. If A, B, and C are identical except that the ith row (column) of C is the sum of the ith rows (columns) of A and B, then $\det C = \det A + \det B$.
f. If B is obtained by adding a multiple of one row (column) of A to another row (column), then $\det B = \det A$.

Proof

(c) If A has two identical rows, swap them to obtain the matrix B. Clearly, $B = A$, so $\det B = \det A$. On the other hand, by (b), $\det B = -\det A$. Therefore, $\det A = -\det A$, so $\det A = 0$.

(d) Suppose row i of A is multiplied by k to produce B; that is, $b_{ij} = ka_{ij}$ for $j = 1, \ldots, n$. Since the cofactors C_{ij} of the elements in the ith rows of A and B are identical (why?), expanding along the ith row of B gives

$$\det B = \sum_{j=1}^{n} b_{ij}C_{ij} = \sum_{j=1}^{n} ka_{ij}C_{ij} = k\sum_{j=1}^{n} a_{ij}C_{ij} = k \det A$$

(e) As in (d), the cofactors C_{ij} of the elements in the ith rows of A, B, and C are identical. Moreover, $c_{ij} = a_{ij} + b_{ij}$ for $j = 1, \ldots, n$. We expand along the ith row of C to obtain

$$\det C = \sum_{j=1}^{n} c_{ij}C_{ij} = \sum_{j=1}^{n} (a_{ij} + b_{ij})C_{ij} = \sum_{j=1}^{n} a_{ij}C_{ij} + \sum_{j=1}^{n} b_{ij}C_{ij} = \det A + \det B$$

Example 4.13 Compute det A if

(a) $$A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 5 & 3 \\ -4 & -6 & 2 \end{bmatrix}$$

(b) $$A = \begin{bmatrix} 0 & 2 & -4 & 5 \\ 3 & 0 & -3 & 6 \\ 2 & 4 & 5 & 7 \\ 5 & -1 & -3 & 1 \end{bmatrix}$$

Theorem 4.3 Let $A = [a_{ij}]$ be a square matrix.

a. If A has a zero row (column), then det $A = 0$.
b. If B is obtained by interchanging two rows (columns) of A, then det $B = -$det A.
c. If A has two identical rows (columns), then det $A = 0$.
d. If B is obtained by multiplying a row (column) of A by k, then det $B = k$ det A.
e. If A, B, and C are identical except that the ith row (column) of C is the sum of the ith rows (columns) of A and B, then det $C =$ det $A +$ det B.
f. If B is obtained by adding a multiple of one row (column) of A to another row (column), then det $B =$ det A.

Definition An ***elementary matrix*** is any matrix that can be obtained by performing an elementary row operation on an identity matrix.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 5 & 7 \\ -1 & 0 \\ 8 & 3 \end{bmatrix} \qquad EA = \begin{bmatrix} 5 & 7 \\ 8 & 3 \\ -1 & 0 \end{bmatrix}$$

Theorem 4.4 Let E be an $n \times n$ elementary matrix.

a. If E results from interchanging two rows of I_n, then $\det E = -1$.
b. If E results from multiplying one row of I_n by k, then $\det E = k$.
c. If E results from adding a multiple of one row of I_n to another row, then $\det E = 1$.

Proof Since $\det I_n = 1$, applying (b), (d), and (f) of Theorem 4.3 immediately gives (a), (b), and (c), respectively, of Theorem 4.4.

Lemma 4.5 Let B be an $n \times n$ matrix and let E be an $n \times n$ elementary matrix. Then

$$\det(EB) = (\det E)(\det B)$$

Theorem 4.3

Let $A = [a_{ij}]$ be a square matrix.

a. If A has a zero row (column), then $\det A = 0$.
b. If B is obtained by interchanging two rows (columns) of A, then $\det B = -\det A$.
c. If A has two identical rows (columns), then $\det A = 0$.
d. If B is obtained by multiplying a row (column) of A by k, then $\det B = k \det A$.
e. If A, B, and C are identical except that the ith row (column) of C is the sum of the ith rows (columns) of A and B, then $\det C = \det A + \det B$.
f. If B is obtained by adding a multiple of one row (column) of A to another row (column), then $\det B = \det A$.

Theorem 4.6
A square matrix A is invertible if and only if $\det A \neq 0$.

Proof Let A be an $n \times n$ matrix and let R be the reduced row echelon form of show first that $\det A \neq 0$ if and only if $\det R \neq 0$.

Summary of the "Dead End"

If R were **not** the identity matrix I_n:

- It would have fewer than n pivots.
- That would mean it has at least one row of zeros.
- That would mean $\det R = 0$.
- That would mean $\det A = 0$.

Since we started with the assumption that $\det A \neq 0$, the only logical conclusion left standing is that **R must be I_n**.

Let $E_1, E_2, \ldots, E_r$ be the elementary matrices corresponding to the elementary row operations that reduce A to R. Then

$$E_r \cdots E_2 E_1 A = R$$

Lemma 4.5 Let B be an $n \times n$ matrix and let E be an $n \times n$ elementary matrix. Then

$$\det(EB) = (\det E)(\det B)$$

applying Lemma 4.5, we obtain

$$(\det E_r) \cdots (\det E_2)(\det E_1)(\det A) = \det R$$

Theorem 4.4 Let E be an $n \times n$ elementary matrix.

a. If E results from interchanging two rows of I_n, then $\det E = -1$.
b. If E results from multiplying one row of I_n by k, then $\det E = k$.
c. If E results from adding a multiple of one row of I_n to another row, then $\det E = 1$.

By Theorem 4.4, the determinants of all the elementary matrices are nonzero.

conclude that $\det A \neq 0$ if and only if $\det R \neq 0$.

Theorem 3.12 **The Fundamental Theorem of Invertible Matrices: Version 1**

Let A be an $n \times n$ matrix. The following statements are equivalent:

a. A is invertible.
b. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.
c. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
d. The reduced row echelon form of A is I_n.
e. A is a product of elementary matrices.

Now suppose that A is invertible.
Then, by the Fundamental Theorem of Invertible Matrices, $R = I_n$, so $\det R = 1 \neq 0$. Hence, $\det A \neq 0$ also.

Theorem 4.3 Let $A = [a_{ij}]$ be a square matrix.

a. If A has a zero row (column), then $\det A = 0$.

Conversely, if $\det A \neq 0$, then $\det R \neq 0$, so R cannot contain a zero row, by Theorem 4.3(a).

It follows that R must be I_n (why?), so A is invertible, by the Fundamental Theorem again.

Theorem 4.7 If A is an $n \times n$ matrix, then

$$\det(kA) = k^n \det A$$

Theorem 4.8 If A and B are $n \times n$ matrices, then

$$\det(AB) = (\det A)(\det B)$$

Proof

If A is invertible, then, by the Fundamental Theorem of Invertible Matrices, elementary matrices $E_1E_2\cdots E_k$

$$A = E_1E_2\cdots E_k$$

Then $AB = E_1E_2\cdots E_kB$, so k applications of Lemma 4.5 give

$$\det(AB) = \det(E_1E_2\cdots E_kB) = (\det E_1)(\det E_2)\cdots(\det E_k)(\det B)$$

Continuing to apply Lemma 4.5, we obtain

$$\det(AB) = \det(E_1E_2\cdots E_k)\det B = (\det A)(\det B)$$

if A is not invertible, then neither is AB, by Exercise 47 in Section 3.3. Thus, by Theorem 4.6, $\det A = 0$ and $\det(AB) = 0$. Consequently, $\det(AB) = (\det A)(\det B)$, since both sides are zero.

Example 4.14

Applying Theorem 4.8 to $A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}$, we find that

$$AB = \begin{bmatrix} 12 & 3 \\ 16 & 5 \end{bmatrix}$$

and that $\det A = 4$, $\det B = 3$, and $\det(AB) = 12 = 4 \cdot 3 = (\det A)(\det B)$, as claimed. (Check these assertions!)

Theorem 4.9 If A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det A}$$

Proof Since A is invertible, $AA^{-1} = I$, so $\det(AA^{-1}) = \det I = 1$. Hence, $(\det A)(\det A^{-1}) = 1$, by Theorem 4.8, and since $\det A \neq 0$ (why?), dividing by $\det A$ yields the result.

Theorem 4.10 For any square matrix A,

$$\det A = \det A^T$$

For an $n \times n$ matrix A and a vector $\mathbf{b}$ in $\mathbb{R}^n$, let $A_i(\mathbf{b})$ denote the matrix obtained by replacing the ith column of A by $\mathbf{b}$.

$$A_i(\mathbf{b}) = [\mathbf{a}_1 \cdots \underset{\text{Column } i}{\mathbf{b}} \cdots \mathbf{a}_n]$$

Theorem 1.11 **Cramer's Rule**

Let A be an invertible $n \times n$ matrix and let $\mathbf{b}$ be a vector in $\mathbb{R}^n$. Then the unique solution $\mathbf{x}$ of the system $A\mathbf{x} = \mathbf{b}$ is given by

$$x_i = \frac{\det(A_i(\mathbf{b}))}{\det A} \quad \text{for } i = 1, \ldots, n$$

Example 4.16

Use Cramer's Rule to solve the system

$$\begin{aligned} x_1 + 2x_2 &= 2 \\ -x_1 + 4x_2 &= 1 \end{aligned}$$

Theorem 4.12 Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A}\operatorname{adj} A$$

the inverse of A is the *transpose* of the matrix of cofactors of A, divided by the determinant of A.

formula for the inverse of a matrix in terms of determinants.

If A is an invertible $n \times n$ matrix, its inverse is the (unique) matrix X that satisfies the equation $AX = I$. Solving for X one column at a time, let $\mathbf{x}_j$ be the jth column of X.

$$\mathbf{x}_j = \begin{bmatrix} x_{1j} \\ \vdots \\ x_{ij} \\ \vdots \\ x_{nj} \end{bmatrix}$$

Therefore, $A\mathbf{x}_j = \mathbf{e}_j$, and by Cramer's Rule, $x_{ij} = \dfrac{\det(A_i(\mathbf{e}_j))}{\det A}$

adjoint (or ***adjugate***) of A and is denoted by adj A.

$$[C_{ji}] = [C_{ij}]^T = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

ith column ↓

$$\det(A_i(\mathbf{e}_j)) = \begin{vmatrix} a_{11} & a_{12} & \cdots & 0 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \iddots & \vdots \\ a_{j1} & a_{j2} & \cdots & 1 & \cdots & a_{jn} \\ \vdots & \vdots & \iddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 & \cdots & a_{nn} \end{vmatrix} = (-1)^{j+i}\det A_{ji} = C_{ji}$$

which is the (j, i)-cofactor of A.

It follows that $x_{ij} = (1/\det A)C_{ji}$, so $A^{-1} = X = (1/\det A)\,[C_{ji}] = (1/\det A)\,[C_{ij}]^T$.

Example 4.17

Use the adjoint method to compute the inverse of

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}$$

Lemma 4.13 Let A be an $n \times n$ matrix. Then

$$a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n} = \det A = a_{11}C_{11} + a_{21}C_{21} + \cdots + a_{n1}C_{n1} \tag{7}$$

Lemma 4.14 Let A be an $n \times n$ matrix and let B be obtained by interchanging any two rows (columns) of A. Then

$$\det B = -\det A$$

Exercises 4.2

5, 9, 12, 17, 23, 26, 29, 37, 48, 49, 56, 59, 61