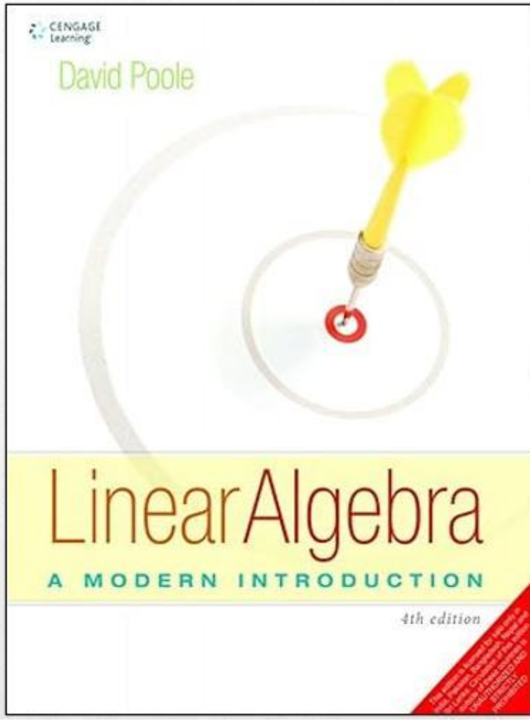

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線性代數 (二)

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4.1 Introduction to Eigenvalues and Eigenvectors

https://hmwu.idv.tw

In Chapter 3,
two applications: Markov chains and the Leslie model of population growth.

For a Markov chain with transition matrix P, a steady state vector $\mathbf{x}$ had the property that

$P\mathbf{x} = \mathbf{x}$;

for a Leslie matrix L, a steady state vector was a population vector $\mathbf{x}$ satisfying

$L\mathbf{x} = r\mathbf{x}$, where r represented the steady state growth rate.

$$\begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 4 & 3 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix} \begin{bmatrix} 18 \\ 6 \\ 1 \end{bmatrix} = 1.5 \begin{bmatrix} 18 \\ 6 \\ 1 \end{bmatrix}$$

for a square matrix A,

we ask whether there exist nonzero vectors $\mathbf{x}$ such that $A\mathbf{x}$ is just a scalar multiple of $\mathbf{x}$. This is the *eigenvalue problem*, and it is one of the most central problems in linear algebra. It has applications throughout mathematics and in many other fields as well.

Definition Let A be an $n \times n$ matrix. A scalar λ is called an ***eigenvalue*** of A if there is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. Such a vector $\mathbf{x}$ is called an ***eigenvector*** of A corresponding to λ.

Example 4.1

Show that $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ and find the corresponding eigenvalue.

Example 4.2

Show that 5 is an eigenvalue of $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ and determine all eigenvectors corresponding to this eigenvalue.

Definition Let A be an $m \times n$ matrix. The ***null space*** of A is the subspace of $\mathbb{R}^n$ consisting of solutions of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. It is denoted by null(A).

Definition Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. The collection of all eigenvectors corresponding to λ, together with the zero vector, is called the ***eigenspace*** of λ and is denoted by E_λ.

Therefore, in Example 4.2, $E_5 = \left\{ t \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

Example 4.3

Show that $\lambda = 6$ is an eigenvalue of $A = \begin{bmatrix} 7 & 1 & -2 \\ -3 & 3 & 6 \\ 2 & 2 & 2 \end{bmatrix}$ and find a basis for its eigenspace.

In $\mathbb{R}^2$, we can give a geometric interpretation of the notion of an eigenvector. The equation $A\mathbf{x} = \lambda\mathbf{x}$ says that the vectors $A\mathbf{x}$ and $\mathbf{x}$ are parallel. Thus, $\mathbf{x}$ is an eigenvector of A if and only if A transforms $\mathbf{x}$ into a parallel vector [or, equivalently, if and only if $T_A(\mathbf{x})$ is parallel to $\mathbf{x}$, where T_A is the matrix transformation corresponding to A].

Example 4.4 Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ geometrically.

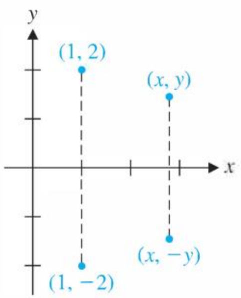

Figure 3.4
Reflection in the x-axis

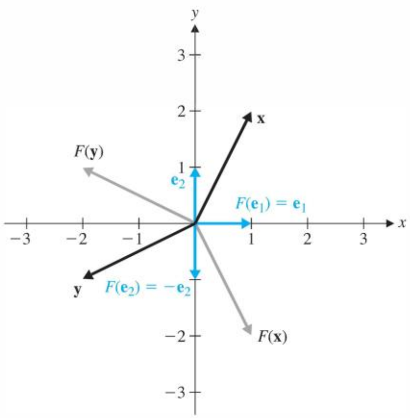

Figure 4.5
The eigenvectors of a reflection

Another way to think of eigenvectors geometrically is to draw $\mathbf{x}$ and $A\mathbf{x}$ head-to-tail. Then $\mathbf{x}$ will be an eigenvector of A if and only if $\mathbf{x}$ and $A\mathbf{x}$ are aligned in a straight line. In Figure 4.6, $\mathbf{x}$ is an eigenvector of A but $\mathbf{y}$ is not.

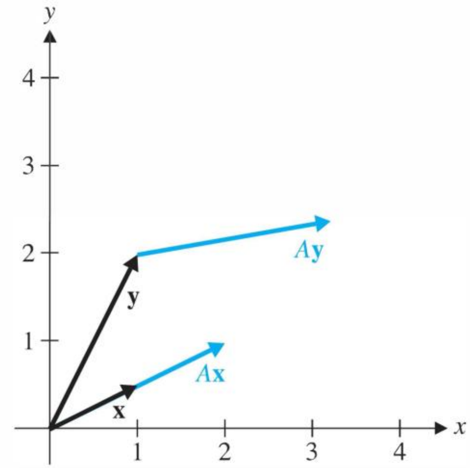

Figure 4.6

We now know how to find eigenvectors once we have the corresponding eigenvalues, and we have a geometric interpretation of them—but one question remains:

How do we first find the eigenvalues of a given matrix?

The key is the observation that λ is an eigenvalue of A if and only if

the null space of $A - \lambda I$ is nontrivial.

Section 3.3

the determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$\det A = ad - bc$, and A is invertible if and only if $\det A$ is nonzero.

Theorem 3.12 **The Fundamental Theorem of Invertible Matrices: Version 1**

Let A be an $n \times n$ matrix. The following statements are equivalent:

a. A is invertible.
b. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.
c. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
d. The reduced row echelon form of A is I_n.
e. A is a product of elementary matrices.

Furthermore, the Fundamental Theorem of Invertible Matrices guarantees that a matrix has a nontrivial null space if and only if it is noninvertible

hence, if and only if its determinant is zero.

λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$.

Example 4.5

Find all of the eigenvalues and corresponding eigenvectors of the matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ from Example 4.1.

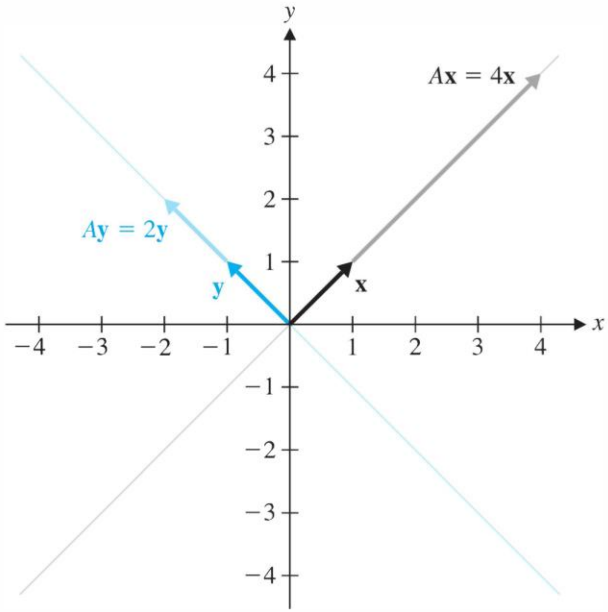

Figure 4.8
How A transforms eigenvectors

an eigenvector $\mathbf{x}$ in the eigenspace E_4
is transformed into $4\mathbf{x}$,
an eigenvector $\mathbf{y}$ in the eigenspace E_2
is transformed into $2\mathbf{y}$.

Example 4.7

Find the eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (a) over $\mathbb{R}$ and (b) over the complex numbers $\mathbb{C}$.

$\mathbb{Z}_p$ (where p is prime) refers to the set of **integers modulo** p .

$$\mathbb{Z}_p = \{0, 1, 2, \ldots, p-1\}$$

2. How it Works: Modular Arithmetic

In $\mathbb{Z}_p$, calculations are performed using **modular arithmetic** (sometimes called "clock arithmetic"). You calculate the result normally, then **divide by** p **and take the remainder**.

Example with $p = 7$ ($\mathbb{Z}_7$):

- $3 + 5 = 8$. Divide 8 by 7, the remainder is **1**. So in $\mathbb{Z}_7$, $3 + 5 = 1$.
- $4 \times 4 = 16$. 16 divided by 7 leaves a remainder of **2** (since $7 \times 2 = 14$). So $4 \times 4 = 2$.

Exercises 4.1

2, 5, 8, 11, 14, 17, 19, 23, 35