Linear Algebra Exam 1: Solution

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1 Multiple choice question

1 B

2 C

3 B

4 B

5 C

2 short answer question

(10%) For a $n \times n$ matrix A,the key point whether A is diagonalizable or not is if A has n linear independent eigenvectors.Because if A has n linear independent eigenvectors $v_1, v_2, \cdots, v_n$ then these eigenvectors can form a invertible matrix $P = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$ such that

$$P^{-1}AP = \text{diag}(\lambda_1, \ldots, \lambda_n)$$

That is A is similar to a diagonal matrix, so A is diagonalizable.
Conversely, if A can't find enough linear independent eigenvectors, then P can't be form like this, hence A is not diagonalizable.

(10%)About the situation for multiple roots : have repeat eigenvalues (algebraic multiplicity > 1),it doesn't implies that A is not diagonalizable. The key point is the geometric multiplicity for that eigenvalue :

- For every eigenvalues λ ,all have

$$1 \leq \text{geometric multiplicity} \leq \text{algebraic multiplicity}.$$

- If for all eigenvalues λ , its geometric multiplicity equal to its algebraic multiplicity, then A can find n linear independent eigenvectors, A is diagonalizable.

- If there exist eigenvalues with geometric multiplicity less than algebraic multiplicity then the number of eigenvectors is not enough, A is not diagonalizable.

Therefore, multiple roots is not problem, geometric multiplicity equal algebraic multiplicity for all eigenvalues is the condition to verify whether A is diagonalizable.

3 compute question

Let

$$A = \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix}$$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -4-\lambda & 6 \\ -3 & 5-\lambda \end{bmatrix}\right) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$$

A has eigenvalues 2, -1.(5%)
For $\lambda_1 = 2$,

$$\begin{bmatrix} -6 & 6 & \vdots & 0 \\ -3 & 3 & \vdots & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{bmatrix}$$

The eigenspace for$\lambda_1 = 2$ is:

$$E_2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x = t, y = t \right\} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \text{(5\%)}$$

For $\lambda_2 = -1$,

$$\begin{bmatrix} -3 & 6 & \vdots & 0 \\ -3 & 6 & \vdots & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{bmatrix}$$

The eigenspace for$\lambda = -1$ is:

$$E_{-1} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x = 2t, y = t \right\} = \text{span}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) \text{(5\%)}$$

Hence A is diagonalizable by

$$P^{-1}AP = D \iff A = PDP^{-1}, P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, P^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \text{(10\%)}$$

$$A^9 = PD^9P^{-1} = \text{(5\%)} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^9 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2^9 - 2 & 2^{10} + 2 \\ -2^9 - 1 & 2^{10} + 1 \end{bmatrix} = \begin{bmatrix} -514 & 1026 \\ -513 & 1025 \end{bmatrix} \text{(10\%)}$$

4 prove question

(15%) First, we prove that if A is diagonalizable, then A has n independent eigenvectors.
Suppose $A \sim D$ (diagonal matrix) via $P^{-1}AP = D, AP = PD$.
Let the columns of P be $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n$
let the diagonal entries of D be $\lambda_1, \lambda_2, \cdots, \lambda_n$.

Then $A \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{bmatrix} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$

$$\text{or} \quad \begin{bmatrix} A\mathbf{p}_1 & A\mathbf{p}_2 & \cdots & A\mathbf{p}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{p}_1 & \lambda_2\mathbf{p}_2 & \cdots & \lambda_n\mathbf{p}_n \end{bmatrix} \tag{1}$$

we have $A\mathbf{p}_1 = \lambda_1\mathbf{p}_1, A\mathbf{p}_2 = \lambda_2\mathbf{p}_2, \cdots, A\mathbf{p}_n = \lambda_n\mathbf{p}_n$

which proves that the column vectors of P are eigenvectors of$A$ corresponding eigenvalues are the diagonal entries of D in the same order.
Since P is invertible, its columns are linear independent,by the fundamental theorem of Invertible Matrices.

(15%) Second, we prove the converse direction.
Suppose A has n linear independent eigenvectors $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n$ with corresponding eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, respectively,

$$\text{then} \quad A\mathbf{p}_1 = \lambda_1\mathbf{p}_1, A\mathbf{p}_2 = \lambda_2\mathbf{p}_2, \cdots, A\mathbf{p}_n = \lambda_n\mathbf{p}_n$$

This implies Equation (2) above, which is equivalent to Equation (1).

$$\begin{bmatrix} A\mathbf{p}_1 & A\mathbf{p}_2 & \cdots & A\mathbf{p}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{p}_1 & \lambda_2\mathbf{p}_2 & \cdots & \lambda_n\mathbf{p}_n \end{bmatrix} \tag{2}$$

Consequently, if we take P to be the $n \times n$ matrix with columns $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n$, then Equation (1) becomes $AP = PD$.
Since the columns of P are linear independent, the Fundamental Theorem of Invertible Matrices implies that P is invertible, so $P^{-1}AP = D$;

that is A is diagonalizable.