Linear Algebra Mid-term: Solution

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April 2026

1 Multiple choice question

1 B

2 C

3 B

4 D

5 C

6 D

2 short answer question

7. Suppose there are two decomposition:$\mathbf{v} = \mathbf{w} + \mathbf{w}^{\perp} = \mathbf{w}_1 + \mathbf{w}_1^{\perp}$
then we have $\mathbf{w} - \mathbf{w}_1 = \mathbf{w}_1^{\perp} - \mathbf{w}^{\perp}$(3%)
the left hand side vectors belong to W(because W is subspace, closed under minus), the right hand side vectors belong to $W^{\perp}$. Hence the equal vector exist in W and $W^{\perp}$ at the same time.

$$\text{Because } W \cap W^{\perp} = \{0\} \ (5\%) \Rightarrow \mathbf{w} - \mathbf{w}_1 = \mathbf{w}_1^{\perp} - \mathbf{w}^{\perp} = 0$$

Hence we have the decomposition is unique.(2%)

8. If column vectors of $A = \{u_1, u_2, \cdots, u_n\}$ is linear independent,then we cam get orthogonal set $\{v_1, v_2, \cdots, v_n\}$ by Gram-Schmidt process.
Let $q_i = \frac{v_i}{\|v_i\|}$ then Q is a matrix with column vectors are q_i that is

$$Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}$$

Because column vectors of Q are orthonormal, we have Q is orthogonal matrix.

$$\text{By } Q^TQ = I \Rightarrow R = Q^TA = \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix} \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$$

$$\Rightarrow R_{ij} = q_i^T u_j$$

According to Gram-Schmidt process, we have $v_i \ \cdot u_j = v_i^T u_j = 0 \ \forall i > j$ Hence R is a upper triangular matrix.

If column vectors of A are linear dependent,without loss of generality, assume $u_n = c_1u_1 + \cdots + c_{n-1}u_{n-1}$ for some constant $c_1, \cdots, c_{n-1}$.
Then we have

$$\text{span}\{u_1, u_2, \cdots, u_{n-1}, u_n\} = \text{span}\{u_1, u_2, \cdots, u_{n-1}\} \tag{1}$$

In Gram-Schmidt process, we have

$$\text{span}\{u_1, \cdots, u_i\} = \text{span}\{v_1, \cdots, v_i\} \text{ for } 1 \le i \le n \tag{2}$$

Combine (1) and (2) then we have

$$W = \text{span}\{v_1, \cdots, v_{n-1}\} = \text{span}\{v_1, \cdots, v_n\} \Rightarrow v_n \in W$$

then

$$\{v_1, \cdots, v_n\} \text{ is not linear independent} \Rightarrow \{v_1, \cdots, v_n\} \text{ is not orthogonal set}$$

Hence we can't find enough orthogonal vector, Q isn't exist.
solution 2
(10%) If column vectors of matrix A are linear dependent, it means that in the Gram-Schmidt Process, there is a column vector$\mathbf{x}_i$ will in the first i-1 column vectors span subspace W_{i-1}. When we try to find $\mathbf{x}_i$ orthogonal component to W_{i-1} ,(that is $\mathbf{v}_i = \mathbf{x}_i - proj_{W_{i-1}}(\mathbf{x}_i)$), because $\mathbf{x}_i$ is in that space ,the projection is itself, which cause $\mathbf{v}_i$ become zero vector. Because zero vector can't be normalize, hence we can't construct orthogonal matrix Q simultaneously, it will cause upper triangular matrix R diagonal element exist 0, cause R is not invertible.

3 computing question

9. Suppose $W = \text{span}\{u_1, u_2\}, u_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Note that $u_1 \cdot u_2 = -2 - 2 + 4 = 0$

Hence the orthogonal projection of$v$ onto W is

$$\operatorname{proj}_W(v) = \frac{v \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{v \cdot u_2}{u_2 \cdot u_2} u_2 (10\%) = \frac{1}{9} u_1 + \frac{13}{18} u_2 (5\%) = \begin{bmatrix} \frac{-1}{2} \\ \frac{1}{2} \\ 3 \end{bmatrix} (5\%)$$

10. Suppose $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix}$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 5-\lambda & 0 & 0 \\ 0 & 1-\lambda & 3 \\ 0 & 3 & 1-\lambda \end{bmatrix}\right) = (5-\lambda)(\lambda-4)(\lambda+2)$$

Hence the eigenvalue of A is -2,4,5.(5%)

For $\lambda = -2$, let $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be eigenvector cor $\lambda = -2$

$$\begin{bmatrix} 7 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & 3 & 3 \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v = \begin{bmatrix} 0 \\ t \\ -t \end{bmatrix}, t \in \mathbb{R}$$

Hence the eigenvector of A cor $\lambda = -2$ is $u_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.(2%)

For $\lambda = 4$, let $v_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be eigenvector cor $\lambda = -2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix}, t \in \mathbb{R}$$

Hence the eigenvector of A cor $\lambda = 4$ is $u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.(2%)

For $\lambda = 5$, let $v_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be eigenvector cor $\lambda = 5$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & 3 \\ 0 & 3 & -4 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}, t \in \mathbb{R}$$

Hence the eigenvector of A cor $\lambda = 5$ is $u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.(2%)

Since the eigenvalue are distinct, the corresponding eigenvector are orthogonal.
Let

$$Q = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \text{ where } q_i = \frac{u_i}{||u_i||}$$

$\Rightarrow$ then A is orthogonally diagonalize by $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \end{bmatrix}$ (5%), $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ (4%).

4 proof question

(5%) 11.The spectral Theorem: Let A be an $n \times n$ real matrix.Then A is symmetric if and only if it is orthogonally diagonalizable.

(13%) First ,we prove that if A is orthogonally diagonalizable, then A is symmetric. Suppose A is orthogonally diagonalizable

$$\Rightarrow Q^T AQ = D \text{ for some } Q : \text{orthogonal matrix}, D : \text{diagonal matrix}$$

$$\Rightarrow A = QDQ^T, A^T = (QDQ^T)^T = Q(QD)^T = QD^TQ^T = QDQ^T = A$$

Hence A is symmetric.

Second,we prove if A is symmetric then A is orthogonally diagonalizable.

(12%) The Spectral Theorem allows us to write a real symmetric matrix in the form $A = QDQ^T$,where Q is orthogonal and D is diagonal.

$$\text{The diagonal entries of } D \text{ are just eigenvalues of } \ A,$$

if the columns of Q are the orthonormal vectors $\mathbf{q}_1, \cdots, \mathbf{q}_n$,
then using the column-row representation of the product,we have

$$A = QDQ^T = \begin{bmatrix} \mathbf{q}_1 & \cdots & \mathbf{q}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{q}_1^T \\ \vdots \\ \mathbf{q}_n^T \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 \mathbf{q}_1 & \cdots & \lambda_n \mathbf{q}_n \end{bmatrix} \begin{bmatrix} \mathbf{q}_1^T \\ \vdots \\ \mathbf{q}_n^T \end{bmatrix}$$

$$= \lambda_1 \mathbf{q}_1 \mathbf{q}_1^T + \lambda_2 \mathbf{q}_2 \mathbf{q}_2^T + \cdots + \lambda_n \mathbf{q}_n \mathbf{q}_n^T \quad \textbf{spectral decomposition} \text{ of } A.$$