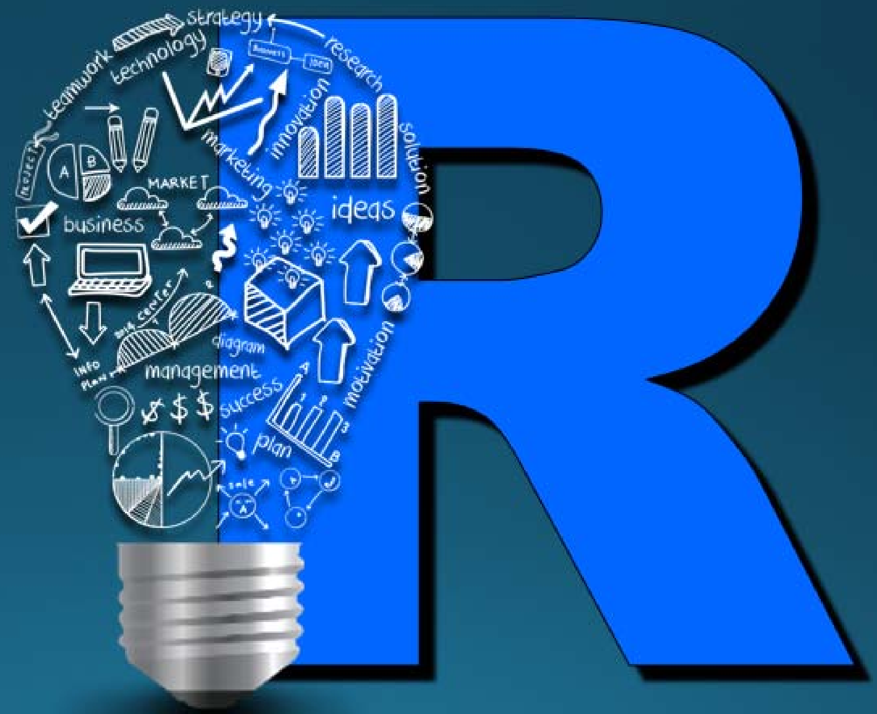
B02

敘述統計
機率分佈與模擬



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http://www.hmwu.idv.tw

本章大綱

- 敘述統計
- 排列組合
- 隨機變數
- 機率分佈
  - 機率質量函數/機率密度函數(d)、累積分佈函數(p)分位數(q)、隨機數(r)
  - 常見之分佈(二項式分佈、常態分佈)、隨機抽樣
- QQplot
- 中央極限定理 (Central Limit Theorem, CLT)
- 用R程式模擬算機率

敘述統計

請舉幾個應用敘述統計於商業、工業、製造業和生物醫學的例子。

當然可以！敘述統計在多個領域中都有廣泛的應用，以下是一些應用敘述統計於商業、工業、製造業和生物醫學的例子：

商業：

1. 市場研究：利用敘述統計對消費者調查數據進行分析，了解消費者的喜好、購買行為和市場趨勢。
2. 銷售分析：分析各個產品或服務的銷售數據，以確定熱銷產品、低銷產品和季節性銷售變化。
3. 客戶滿意度調查：通過對客戶滿意度數據進行敘述統計分析，企業可以找到改進服務品質的方向。

工業：

1. 品質控制：在生產過程中，利用敘述統計分析產品的
標準。
2. 生產效率：通過分析生產線的運行速度、產量等數
措施改進。
3. 能源消耗分析：對工業生產中的能源消耗進行敘述

製造業：

1. 供應鏈管理：通過分析供應商的交貨時間、產品質量等數據，可對供應鏈進行優化。
2. 存貨管理：利用敘述統計分析庫存水平，以確保適時補充並減少過剩庫存造成的損失。
3. 員工生產力：通過對員工生產力數據進行敘述統計分析，找出影響生產力的因素，並制定相應的培訓和激勵措施。

生物醫學：

1. 臨床試驗：在藥物研究和開發過程中，利用敘述統計分析試驗結果，以確定藥物的療效和安全性。
2. 流行病學研究：對疾病發病率、死亡

資料分析的基本流程

Source: 資料科學的統計實務，温政堯 譯，旗標出版社，2021/11/01。

到底要用哪一種處理/分析方法?

統計方法? 大數據? 機器學習? 深度學習? AI?

今日新聞 NOWnews 60.9k 人追蹤 ☆ 追蹤

善用房產科技！靠數位工具突破限制

生活消費中心/綜合報導

2022年11月22日

房地產科技PropTech（Property Technology），結合房地產銷售結合AI人工智慧、大數據、機器學習等數位科技，藉此突破時間和空間限制，提升銷售效率，包括增加消費者和物件的媒合效率，國外已有許多房地產業者逐漸向科技靠攏，而國內房仲龍頭信義房屋近年也致力於PropTech發展，以科技賦能輔助，希望為買賣雙方建構出更多元、貼心的服務。

http://shorturl.at/kzOU7

http://shorturl.at/oGY04

3.1k 人追蹤 ☆ 追蹤 yahoo! 新聞

大數據分析交通違規事件 違停居首

【記者張淑珠／台中報導】

2023年2月7日 週二 下午8:47

在重大違規部分，黃士哲說，依據無照駕駛違規態樣分析，實際未領有駕照有59%，其中未滿18歲占14%、滿18歲占45%；進一步分析，未領有駕照、且未滿18歲所駕駛的車種有95%是機車。對此，交通事件裁決處自110年6月起，針對未滿18歲違規駕駛案，每月挑檔提供清冊、交由警察局進行後續個案關懷協助，警察局另將學籍在台中市的無照駕駛列管案件，每月通報教育局轉各校，並啟動校園關懷輔導。

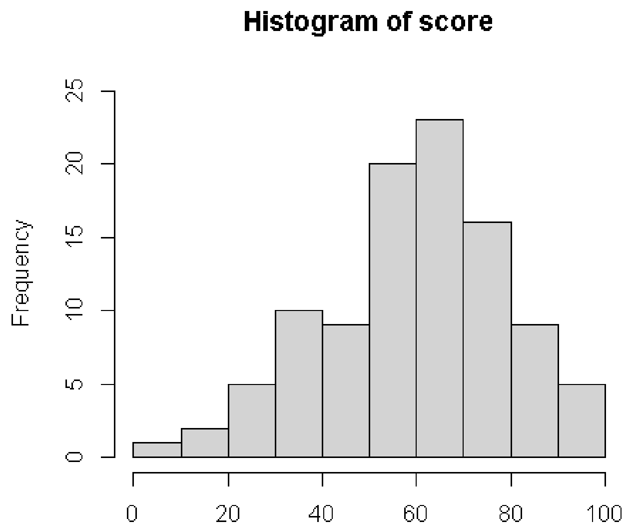
此外，針對酒後駕車，經大數據分析顯示，男性為主要的酒駕違規者，且在各年齡層的每十萬人違規人數中，以成年男性（25-64歲）酒駕情形最為嚴重，年輕男性（18-24歲）次之。交通事件裁決處表示，將強力配合法務部行政執行署執行滯欠酒

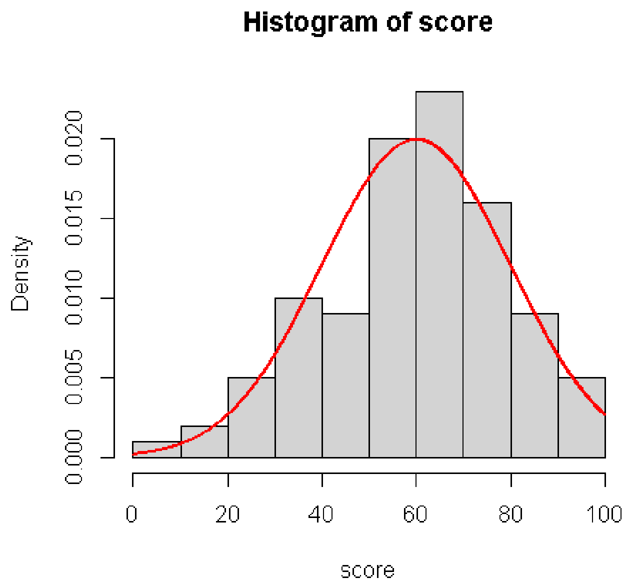
什麼是統計?

- Merriam-Webster dictionary defines statistics as "a branch of mathematics dealing with the collection, analysis, interpretation, and presentation of masses of numerical data."

- 傳統統計(歷史源自17世紀), 分兩類:
  - **敘述統計**: 對所收集到樣本的摘要結果。
  - **推論統計**: 考慮隨機性之下，根據樣本的特性去推論母體的參數(例如: 估計母體平均數、推論母體的分佈)。

- 統計研究領域的分類: 數理統計、工業統計、商用統計、生物統計、社會統計、貝氏統計、空間統計等等。





http://www.theusrus.de/blog/some-truth-about-big-data/

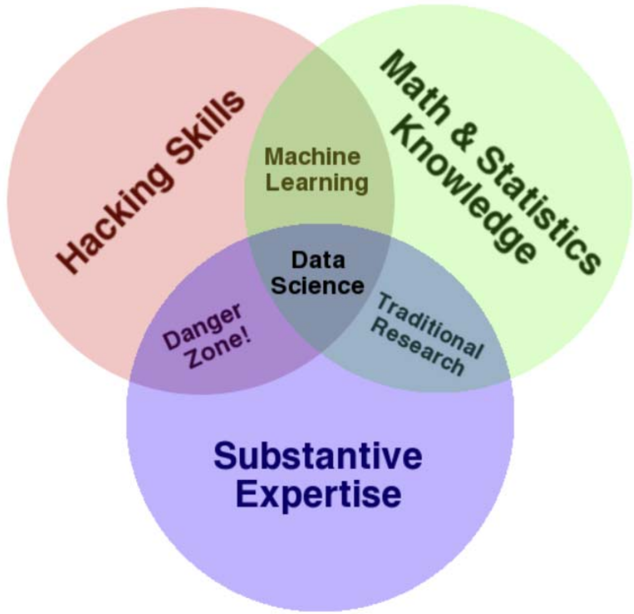
統計、資料探勘、機器學習、資料科學

- **Machine Learning** is an algorithm that can learn from data without relying on rules-based programming.
- **Statistical Modelling** is the formalization of relationships between variables in the form of mathematical equations.

Machine learning	Statistics
network, graphs	model
weights	parameters
learning	fitting
generalization	test set performance
supervised learning	regression/classification
unsupervised learning	density estimation/ clustering

The Data Science Venn Diagram

http://drewconway.com/zia/2013/3/26/the-data-science-venn-diagram



機器學習和統計模型的差異
http://vvar.pixnet.net/blog/post/242048881
為什麼統計學家、機器學習專家解決同一問題的方法差別那麼大?
https://read01.com/EBPPK7.html
機器學習與統計學是互補的嗎？
https://read01.com/ezQ3K.html

資料的型態 (Types of Data Scales)

- Nominal (名目變數), Categorical (類別資料), discrete: 性別、種族、宗教信仰、交通工具、音樂類型... (qualitative 屬質)。

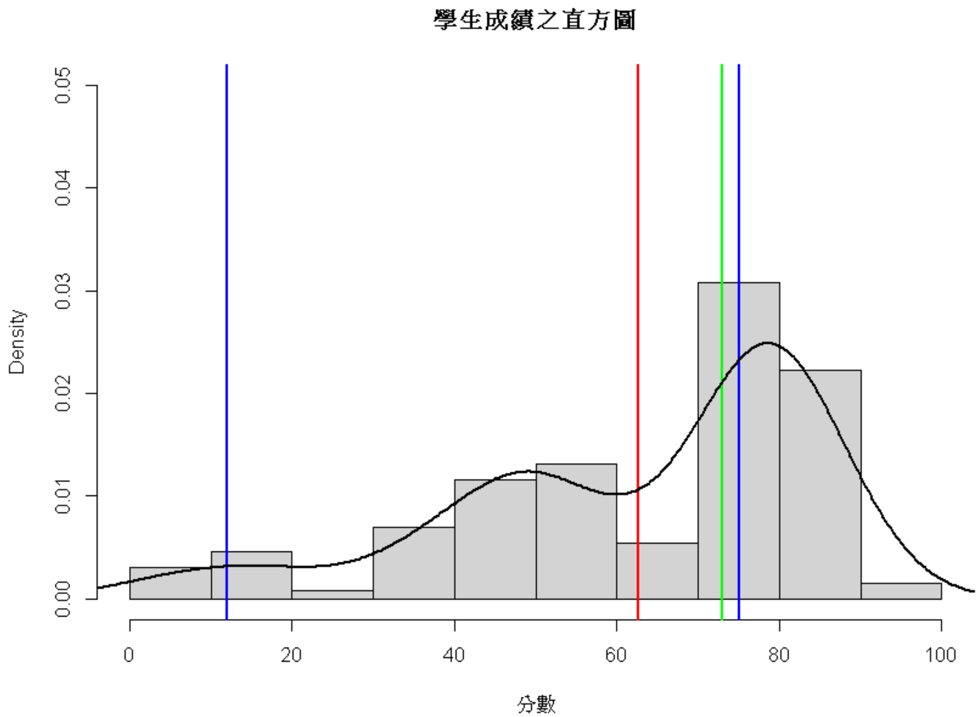
- Ordinal (順序): 精通程度、同意程度、滿意程度、教育程度。

- Interval — Distances between values are meaningful, but **zero point** is not meaningful. (例如:華氏溫度)(不能說：80 度是4 0度的兩倍熱)。

- Ratio (Continuous Data 連續型資料)— Distances are meaningful and a zero point is meaningful: 年收入、年資、身高、... (quantitative 計量)。

資料描述: 中心趨勢、分散程度

- 資料中心趨勢:
  - 平均數(average)
  - 眾數(mode)
  - 中位數(median)



- 資料分散程度:
  - 四分位數(Quartile)
  - 全距(range)
  - 四分位距(interquartile range, IQR)
  - 百位數(percentile)
  - **標準差(standard deviation)**
  - 變異數(variance)
  - **Median absolute deviation**

$$s_x = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}}$$

$n =$ The number of data points

$\bar{x} =$ The mean of the x_i

$x_i =$ Each of the values of the data

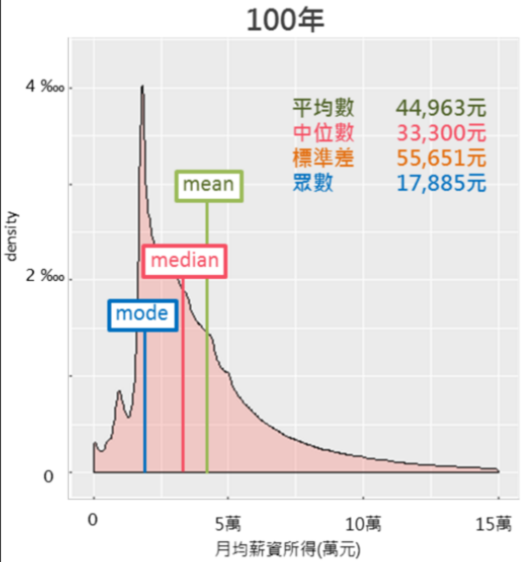
$$\tilde{X} = \text{median}(X):$$

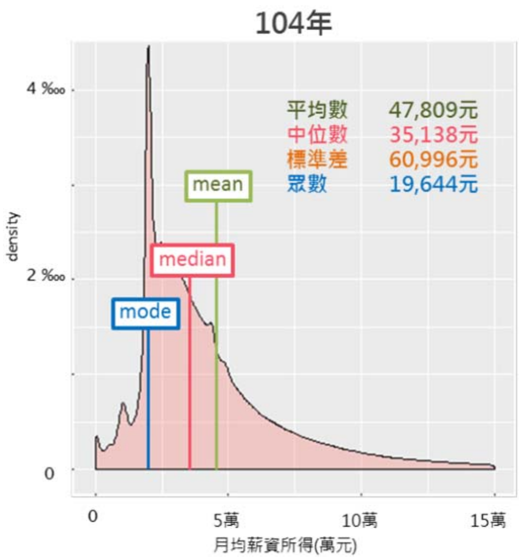
$$\text{MAD} = \text{median}(|X_i - \tilde{X}|)$$

- 偏態係數
- 峰態係數

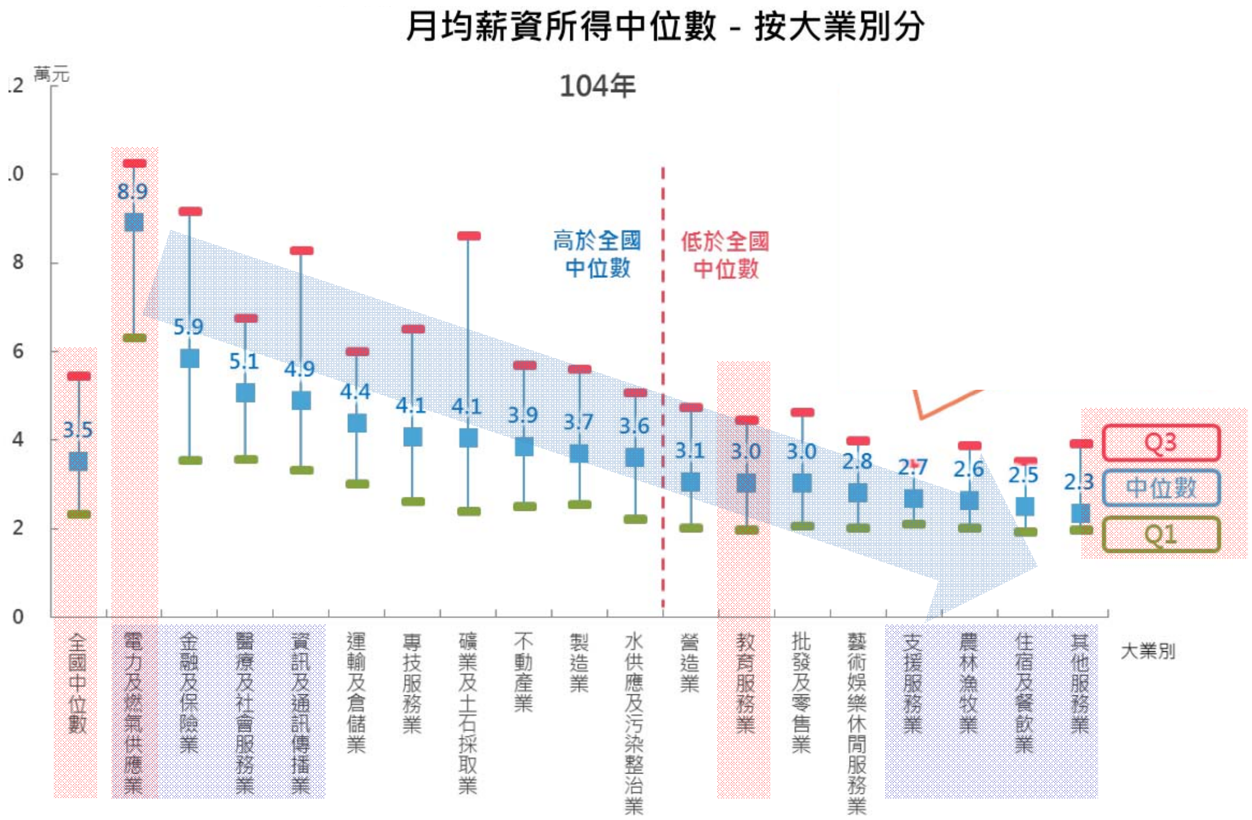
範例: 由財稅大數據探討臺灣近年薪資樣貌

月均薪資所得機率分布圖





由財稅大數據探討臺灣近年薪資樣貌 財政部統計處 106年8月
https://www.mof.gov.tw/File/Attach/75403/File_10649.pdf



玩玩看~薪情平臺

行政院主計總處 Directorate-General of Budget, Accounting and Statistics, Executive Yuan, R.O.C. (Taiwan)　熱誠．公正．效率．精確　統計資訊網　答客問

https://earnings.dgbas.gov.tw/

我的薪資合理嗎？行政院「薪情平台」，一看就知道自己的薪水等級
https://buzzorange.com/techorange/2019/04/10/salary-distribution/

機率分佈

要如何介紹機率分佈(probability distribution)給一般人或數學能力較不好的人了解?

要向
解釋
機率
訴我
機事
以擲
子時
同，

要對一般人教授統計學和資料科學的知識，可以不講述機率分佈嗎?

對於 … 注於一
些更 … 析方法
等。 … 。
然而 … 分佈在
許多 … 讓學生
更全
在教 … 概念。
例如 … 注深入
瞭解

一般來說，機率分佈(probability distribution)應用在哪些場域?

機率分佈在許多場域中都有廣泛的應用，以下是一些常見的應用場域：

1. 統計學與資料科學：在統計學和資料科學中，機率分佈被用來建立概率模型、估計參數、進行假設檢驗等。例如，使用正態分佈進行回歸分析、用泊松分佈分析事件的發生率等。
2. 金融與風險管理：在金融領域，機率分佈被用來分析和預測金融市場的波動、評估投資組合的風險，以及建立風險模型等。
3. 工程與品質控制：在工程領域，機率分佈可以用來分析設備的失效概率、預測維修需求，以及進行可靠性分析等。
4. 電信與網絡流量分析：在電信和網絡領域，機率分佈用於分析和預測通信網絡的流量，並優化網絡資源分配。
5. 生物統計學與醫學研究：在生物統計學和醫學研究中，機率分佈用於分析基因變異、進行臨床試驗分析，以及研究疾病的流行病學等。
6. 機器學習與人工智能：在機器學習和人工智能領域，機率分佈被用來建立概率模型，如隱馬爾可夫模型、貝葉斯網絡等，並用於對未知數據進行推斷和預測。
7.
8.

瞭解和掌握機率分佈有助於在各種情況下更好地分析數據、預測未來事件和做出決策。在現代社會中，機率分佈在幾乎所有學科和領域中都發揮著重要作用，因此瞭解機率分佈對於解決現實問題具有很高的價值。

排列數與組合數

```
> x <- letters[1:3]
> library(combinat)
> permn(x)
[[1]]
[1] "a" "b" "c"

[[2]]
[1] "a" "c" "b"

[[3]]
[1] "c" "a" "b"

[[4]]
[1] "c" "b" "a"

[[5]]
[1] "b" "c" "a"

[[6]]
[1] "b" "a" "c"
> # the number of distinct arrangements of 3 out of 5
> nCm(5, 3)
[1] 10
```

- **combn{utils}**: Generate All Combinations of **n** Elements, Taken **m** at a Time.
- Usage: `combn(x, m, FUN = NULL, simplify = TRUE, ...)`
- Arguments:
  - **x**: vector source for combinations, or integer **n** for `x <- seq_len(n)`.
  - **m**: number of elements to choose.
  - **FUN**: function to be applied to each combination; default NULL means the identity, i.e., to return the combination (vector of length **m**).

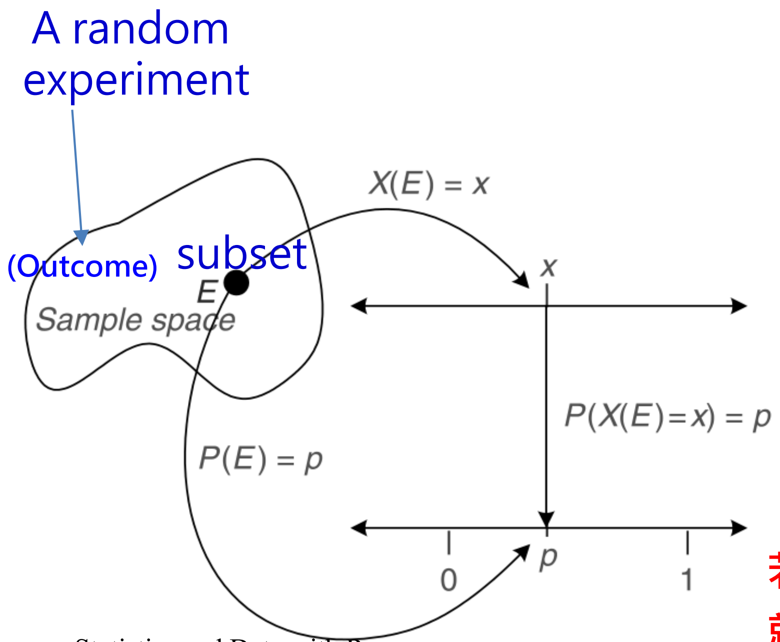
```
> combn(5, 3)
     [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10]
[1,]    1    1    1    1    1    1    2    2    2     3
[2,]    2    2    2    3    3    4    3    3    4     4
[3,]    3    4    5    4    5    5    4    5    5     5
> combn(5, 3, min)
 [1] 1 1 1 1 1 1 2 2 2 3
> choose(5, 3)
[1] 10
```

常見統計名詞

- A **random experiment (隨機實驗)** is a process by which we observe something uncertain. After the experiment, the result of the random experiment is known.
- **Outcome (結果)**: An outcome is a result of a random experiment.
- **Sample space (樣本空間), *S***: the set of all possible outcomes.
  - 例子1: 投擲兩硬幣, 正(Head)反(Tail)面之樣本空間 S={HH, HT, TH, TT}.
- **Event (事件), *E***: an event is a subset of the sample space.
  - 例子2: In the context of an experiment, we may define the sample space of observing a person as S = {sick, healthy, dead} . The following are all events: {sick} , {healthy} , {dead} , {sick, healthy} , {sick, dead} , {healthy, dead} , {sick, healthy, dead} , {none of the above} .
- **Trial (試驗)**: a single performance of an experiment whose outcome is in S.
  - 例子3: 投擲4枚硬幣的隨機實驗中，每投擲一次硬幣皆是一次「試驗」。

機率與隨機變數

- **Probability (機率)**: the probability of event **E**, **P(E)**, is the value approached by the relative frequency of occurrences of **E** in a long series of replications of a random experiment.
  (The frequentist view)
- **Random variable (隨機變數)**: A function that assigns real numbers to events, including the null event.



Source: Statistics and Data with R

Probability Distribution (機率分佈): 是以數學函數的方式來表示隨機實驗中不同的可能結果(即樣本空間之每個元素)發生的可能性(機率)。

例子: 假如令隨機變數 *X* 表示是投擲一枚公平硬幣的結果:
X=1為正面，*X*=0為反面，
則*X*的機率分佈是: P(*X*=1) = 0.5, P(*X*=0) = 0.5.

若將隨機變數每種情況的機率值都列出來，就是其機率分佈。

編註： 所謂機率分佈，就是每個可能出現的值 (例如：1、2、3、4、5、6)，各有多少機率 (例如：$\frac{1}{6}$、$\frac{1}{6}$、$\frac{1}{6}$、$\frac{1}{6}$、$\frac{1}{6}$、$\frac{1}{6}$)，把它們全部列出來，如上表所示。

統計分配 (Statistical Distributions)

Four fundamental items can be calculated for a statistical distribution:

- **機率密度函數值(d)**
  - point probability ***P(X=x)*** or ***probability density function*** *f(x)*: **`dnorm()`**
- **累積機率函數值 (p)**
  - cumulative probability distribution function, *F(x) = P(X<=x)*: **`pnorm()`**
- **分位數 (q)**
  - the quantiles of the distribution: **`qnorm()`**
    Often, we are interested in the inverse of a distribution. That is, given a probability value p, we wish to find the quantile, x, such that $P(X \leq x|\theta) = p$.
- **隨機數 (r)**
  - the random numbers generated from the distribution: **`rnorm()`**

常用機率分配

以常態分佈normal為例:

機率密度(分配)函數: dnorm()

累積機率(分配)函數: pnorm()

分位數: qnorm()

隨機數: rnorm()

Distribution	R name	additional arguments
beta	beta	shape1, shape2, ncp
binomial	binom	size, prob
Cauchy	cauchy	location, scale
chi-squared	chisq	df, ncp
exponential	exp	rate
F	f	df1, df1, ncp
gamma	gamma	shape, scale
geometric	geom	prob
hypergeometric	hyper	m, n, k
log-normal	lnorm	meanlog, sdlog
logistic	logis	location, scale
negative binomial	nbinom	size, prob
normal	norm	mean, sd
Poisson	pois	lambda
Student's	t	t df, ncp
uniform	unif	min, max
Weibull	weibull	shape, scale
Wilcoxon	wilcox	m, n

機率質量函數
Probability Mass Function, PMF

Formal definition

https://en.wikipedia.org/wiki/Probability_mass_function

Suppose that $X: S \to A$ ($A \subseteq \mathbf{R}$) is a discrete random variable defined on a sample space S. Then the probability mass function $f_X: A \to [0, 1]$ for X is defined as

$$f_X(x) = \Pr(X = x) = \Pr(\{s \in S : X(s) = x\}).$$

Thinking of probability as mass helps to avoid mistakes since the physical mass is conserved as is the total probability for all hypothetical outcomes x:

$$\sum_{x \in A} f_X(x) = 1$$

***例子.*投擲2顆公正的骰子**

$X_1 \sim DiscreteUniform\ (1,\ 6).$

$X_2 \sim DiscreteUniform(1,\ 6).$

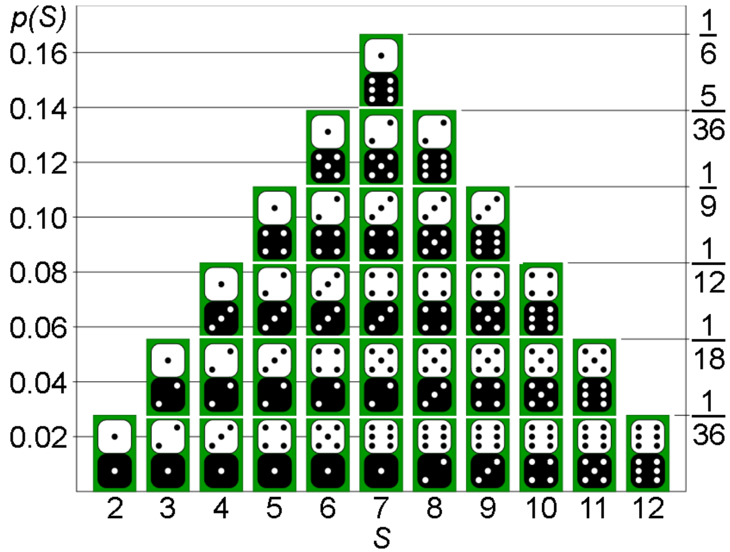
$f_{X1}(k) = f_{X2}(k) = P(X_1 = k) = P(X_2 = k) = 1/6,$
$\quad k = 1,..,6.$

$S = X_1 + X_2$

$f_S(s) = p(S = s),\ s=2,\ ...,\ 12.$

$P(S = 2) = 1/36,\ P(S{=}3)=2/36,\ ...,\ P(S{=}12)=1/36$

$P(X_1 + X_2 > 9) = 1/12 + 1/18 + 1/36 = 1/6$



pmf ($p(S)$) specifies the probability distribution for the sum S of counts from two dice.

https://en.wikipedia.org/wiki/Probability_distribution

機率密度函數
Probability Density Function, PDF

Definition. The **probability density function** ("**p.d.f.**") of a continuous random variable X with support S is an integrable function $f(x)$ satisfying the following:

(1) $f(x)$ is positive everywhere in the support S, that is, $f(x) > 0$, for all x in S

(2) The area under the curve $f(x)$ in the support S is 1, that is: $\int_S f(x)dx = 1$

(3) If $f(x)$ is the p.d.f. of x, then the probability that x belongs to A, where A is some interval, is given by the integral of $f(x)$ over that interval, that is:

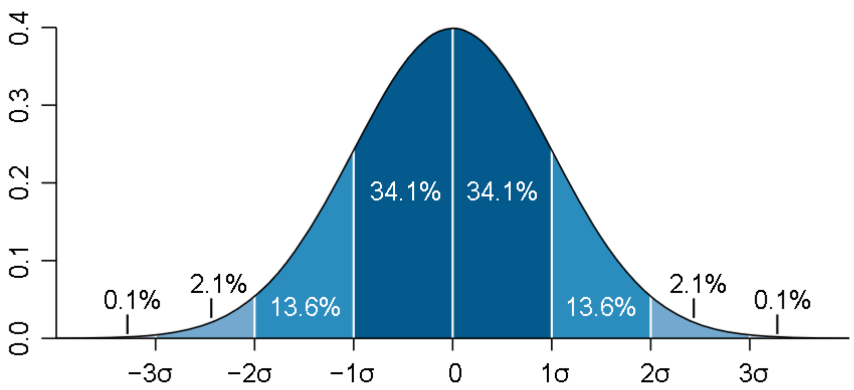
$$P(X \in A) = \int_A f(x)dx \qquad \mathrm{P}[a \le X \le b] = \int_a^b f(x)\,dx$$

The probability density of the normal distribution is:

$$f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where

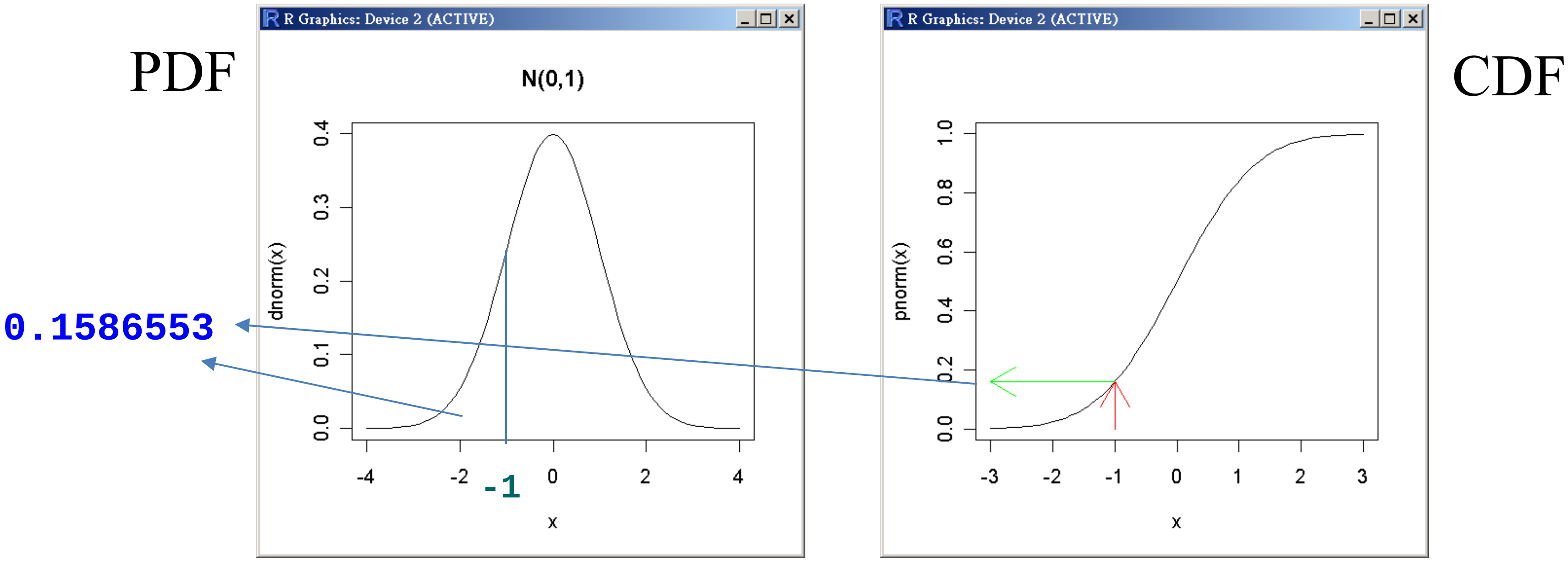
- μ is the mean or expectation of the distribution (and also its median and mode).
- σ is the standard deviation
- σ^2 is the variance



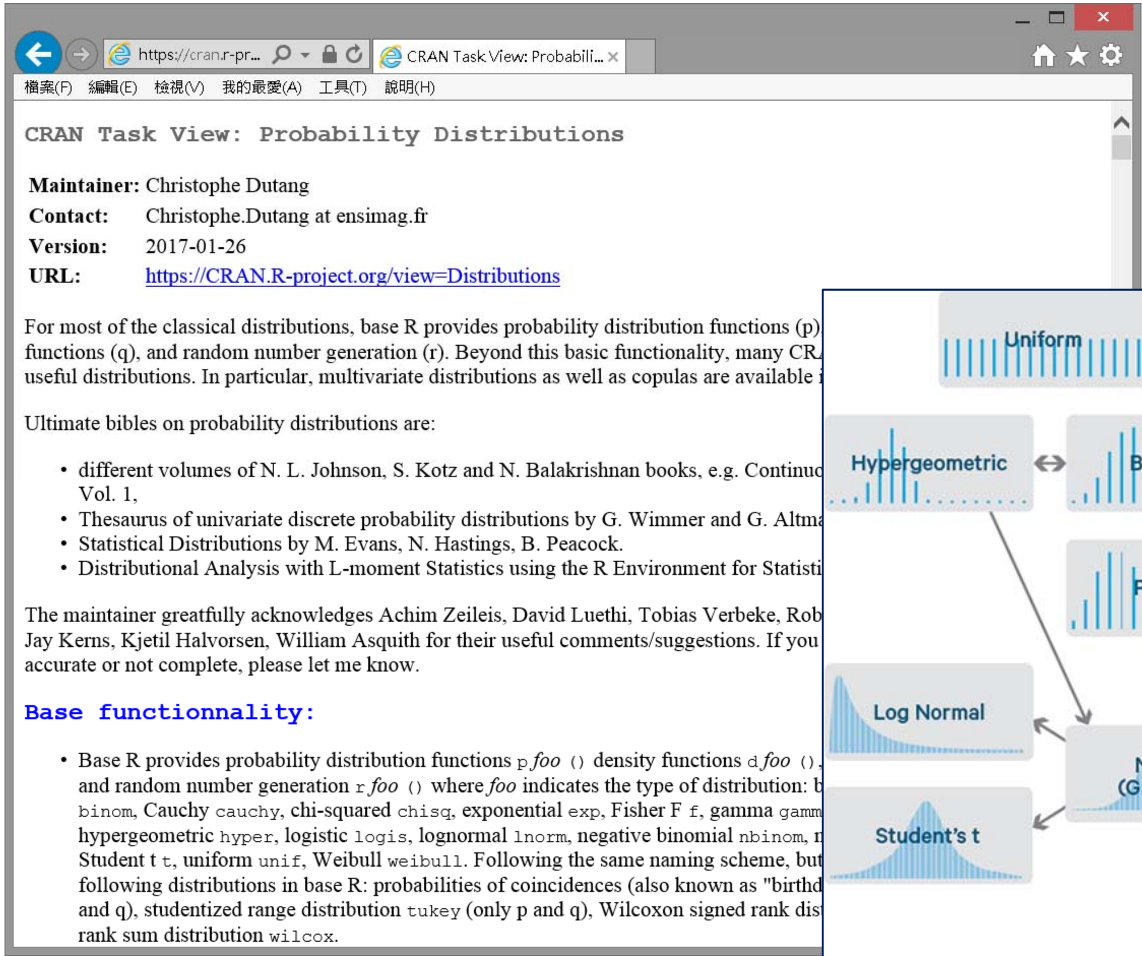
累積機率分配函數 Cumulative Probability Function, CDF

- It is an S-shaped curve showing for any value of x, the probability of obtaining a sample value that is less than or equal to ***x***, ***P(X<= x)***.
- The probability density is the slope of this curve (its derivative) of the cumulative probability function.

```
> curve(pnorm(x), -3, 3)
> arrows(-1, 0, -1, pnorm(-1), col="red")
> arrows(-1, pnorm(-1), -3, pnorm(-1), col="green")
> pnorm(-1)
[1] 0.1586553
```



CRAN Task View: Probability Distribution


https://cran.r-project.org ... CRAN Task View: Probabili...

檔案(F) 編輯(E) 檢視(V) 我的最愛(A) 工具(T) 說明(H)

CRAN Task View: Probability Distributions

Maintainer: Christophe Dutang
Contact: Christophe.Dutang at ensimag.fr
Version: 2017-01-26
URL: https://CRAN.R-project.org/view=Distributions

For most of the classical distributions, base R provides probability distribution functions (p) functions (q), and random number generation (r). Beyond this basic functionality, many CR useful distributions. In particular, multivariate distributions as well as copulas are available i

Ultimate bibles on probability distributions are:

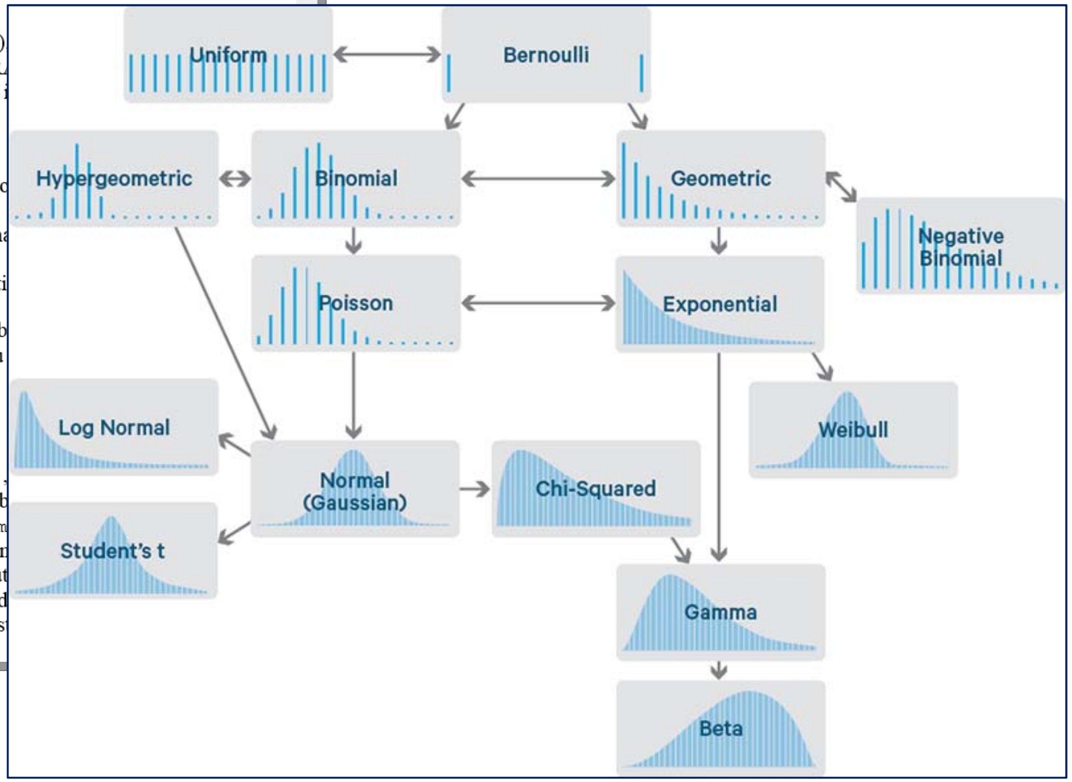
- different volumes of N. L. Johnson, S. Kotz and N. Balakrishnan books, e.g. Continu Vol. 1,
- Thesaurus of univariate discrete probability distributions by G. Wimmer and G. Altm
- Statistical Distributions by M. Evans, N. Hastings, B. Peacock.
- Distributional Analysis with L-moment Statistics using the R Environment for Statisti

The maintainer greatfully acknowledges Achim Zeileis, David Luethi, Tobias Verbeke, Rob Jay Kerns, Kjetil Halvorsen, William Asquith for their useful comments/suggestions. If you accurate or not complete, please let me know.

Base functionality:

- Base R provides probability distribution functions p*foo* () density functions d*foo* () and random number generation r*foo* () where *foo* indicates the type of distribution: b binom, Cauchy cauchy, chi-squared chisq, exponential exp, Fisher F f, gamma gamm hypergeometric hyper, logistic logis, lognormal lnorm, negative binomial nbinom, n Student t t, uniform unif, Weibull weibull. Following the same naming scheme, but following distributions in base R: probabilities of coincidences (also known as "birthd and q), studentized range distribution tukey (only p and q), Wilcoxon signed rank dis rank sum distribution wilcox.

https://cran.r-project.org/web/views/Distributions.html



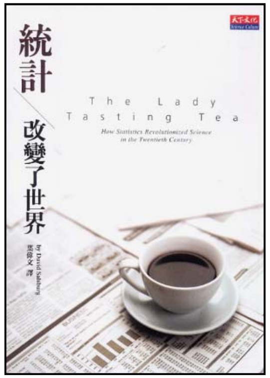
http://blog.cloudera.com/blog/2015/12/common-probability-distributions-the-data-scientists-crib-sheet/

Univariate Distribution Relationships: http://www.math.wm.edu/~leemis/chart/UDR/UDR.html

機率分佈在統計學中的重要性

統計改變了世界

- 十九世紀初:「機械式宇宙」的哲學觀
- 二十世紀: 科學界的統計革命。
- 二十一世紀: 幾乎所有的科學已經轉而運用統計模式了。



統計革命的起點

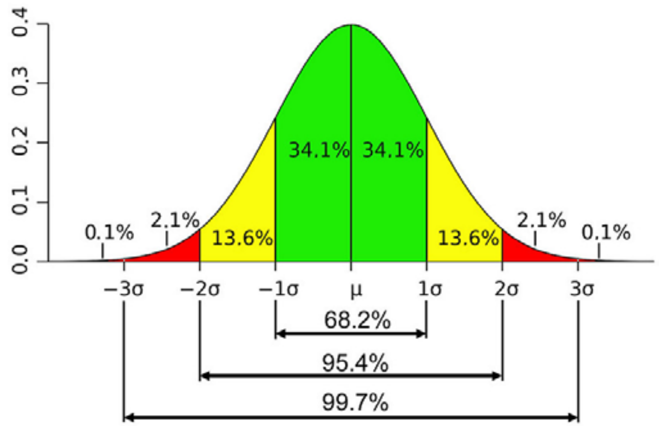
- 1895-1898，發表一系列和相關性(correlation) 有關的論文，涉及動差、相關係數、標準差、卡方適合度檢定，奠定了現代統計學的基礎。
- 引入了統計模型的觀念: 如果能夠決定所觀察現象的**機率分佈的參數**，就可以了解所觀察現象的本質。

樣本變異數與樣本標準差

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2$$

母體變異數與母體標準差

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(X_i - \mu)^2$$



Schweizer, B. (1984), Distributions Are the Numbers of the Future, in Proceedings of The Mathematics of Fuzzy Systems Meeting, eds. A. di Nola and A. Ventre, Naples, Italy: University of Naples, 137–149. (The present is that future.)

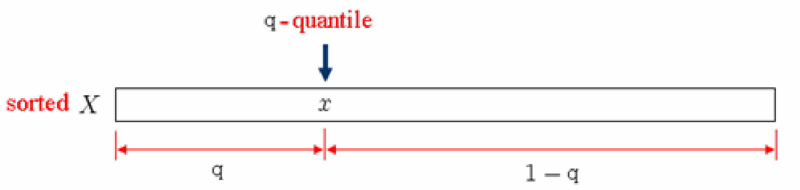
常用機率分佈的應用

- **Normal distribution**, for a single real-valued quantity that grow linearly (e.g. errors, offsets)
- **Log-normal distribution**, for a single positive real-valued quantity that grow exponentially (e.g. prices, incomes, populations)
- **Discrete uniform distribution**, for a finite set of values (e.g. the outcome of a fair die)
- **Binomial distribution**, for the number of "positive occurrences" (e.g. successes, yes votes, etc.) given a fixed total number of independent occurrences
- **Negative binomial distribution**, for binomial-type observations but where the quantity of interest is the number of failures before a given number of successes occurs.
- **Chi-squared distribution**, the distribution of a sum of squared standard normal variables; useful e.g. for inference regarding the sample variance of normally distributed samples.
- **F-distribution**, the distribution of the ratio of two scaled chi squared variables; useful e.g. for inferences that involve comparing variances or involving R-squared.

https://en.wikipedia.org/wiki/Probability_distribution

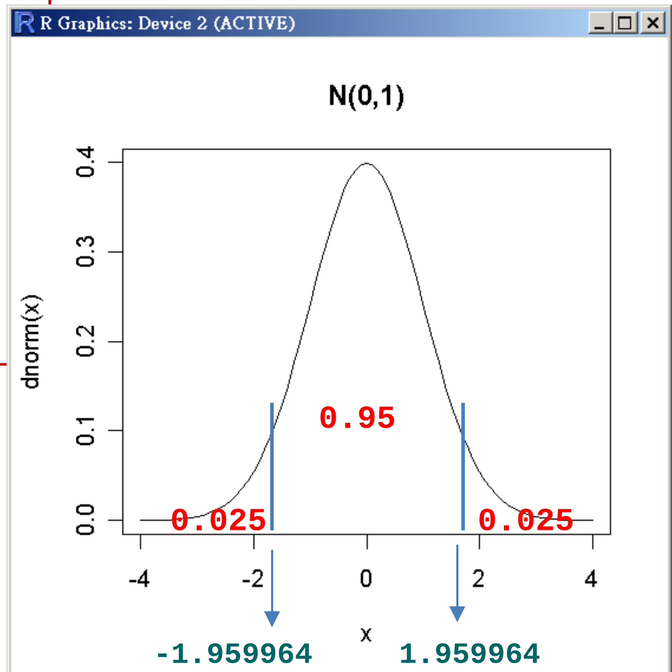
分位數 Quantiles (q)

- The quantile function is the inverse of the cumulative distribution function: $F^{-1}(p) = x$.
- We say that q is the $x\%$-quantile if $x\%$ of the data values are $\leq q$.



$P(X < x) \leq q$ and $P(X > x) \leq 1 - q$.

```
> # 2.5% quantile of N(0, 1)
> qnorm(0.025)
[1] -1.959964
> # the 50% quantile (the median) of N(0, 1)
> qnorm(0.5)
[1] 0
> qnorm(0.975)
[1] 1.959964
```



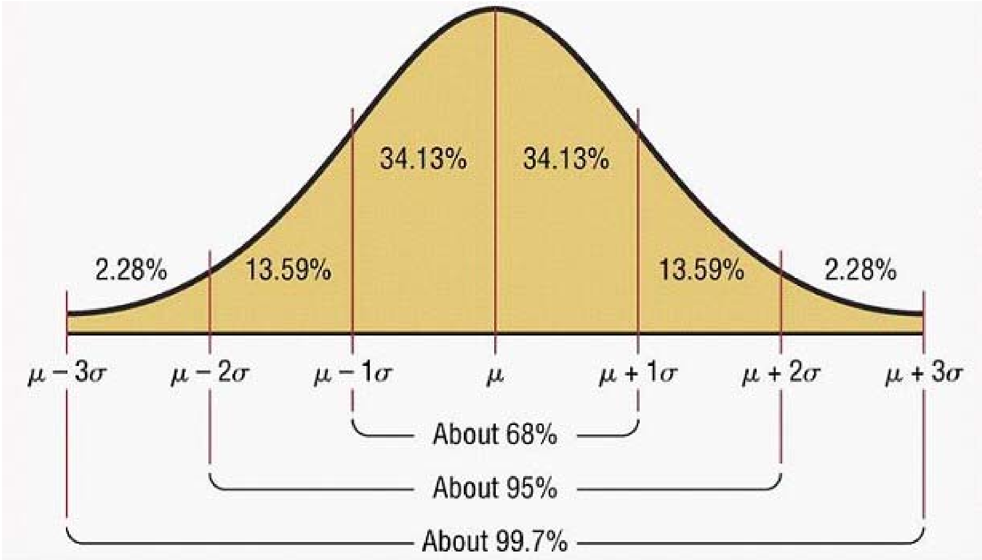
$$\Phi^{-1}(0.975)$$

$$\bar{x} + z_{0.025}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + z_{0.975}\frac{\sigma}{\sqrt{n}}$$

$$P(z_{0.025} \leq \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \leq z_{0.975}) = 0.95$$

範例: 六標準差管理 (Six Sigma, 6σ)

- 標準差是用來描述母體的「變異性」(variation)或「不一致程度」(inconsistency)。
- 過去習慣用「平均數」作為其績效表現，例如平均成本、平均產能、平均交貨時間、平均工資等，但卻忽略了變異程度。現今用「標準差」來衡量產品之品質分佈的變異狀況。
- 六標準差管理是1986年Motorola發展出來的管理方法。符合6σ就代表每生產出一百萬個產品，其不良品必須低於3.4個(良率99.99966%)。轉換成一般服務業的用語，就是每一百萬人次的客人，其中不滿意服務的不可以超過四人。



Source: https://www.usastock88.com/2013/04/BBand.html

- **在美國而言(Harry「1978)，平均99%的品質水準相當於 (假設資料呈現常態分佈)：**
  (1)每小時有2萬件郵件遺失，這相當於1%的誤投率。(2)每天供應的自來水有15分鐘是不適合飲用的。(3)每星期有5千例外科誤診。(4)每個月有7小時停電。

中華六標準差管理學會
https://www.6sigmai.org/

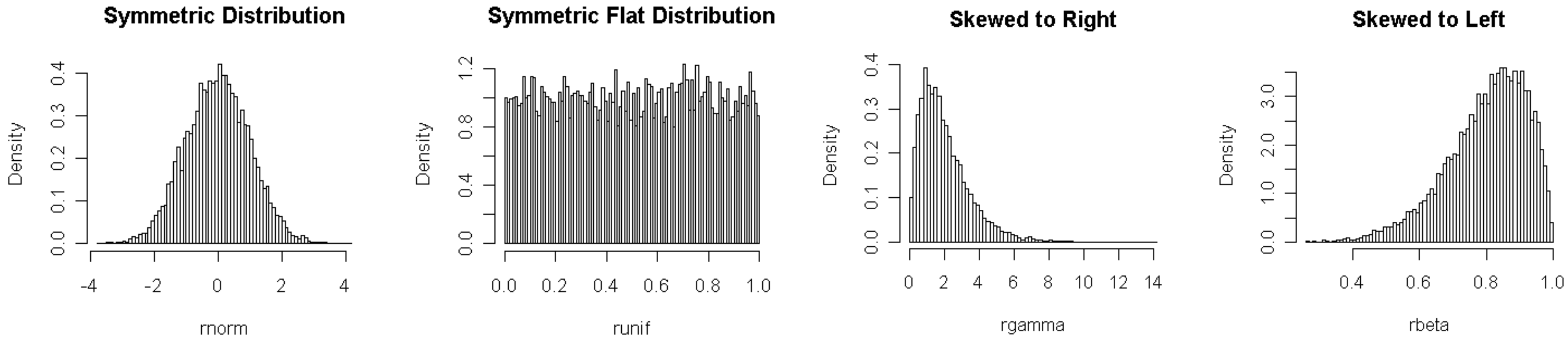
- **對良率的要求需更嚴格的場景:**
  (1) 某航空公司宣稱其飛安率可高達99.73%。(每起落一百萬個架次，約有2,700次失事)
  (2) 某醫院聲明其婦產科接生新生嬰兒的平均成功率是99.9%。

- 六標準差是一個利用統計、問題排除和問題預防等工具，將客戶滿意度提升至99.999%完美層級的品質改善商業策略。

隨機數 Random Numbers (r)

- *Let* X_i is a vector of measurements for the *i*-th object in the sample.
- $(X_1, X_2, \ldots, X_n)$ is said to be a random sample of size *n* from the common distribution if $X_1, X_2, \ldots, X_n$ as independent copies of an underlying measurement vector. (an *n*-tuple of identically-distributed independent random variables).

```
> par(mfrow=c(2,2))
> hist.sym <- hist(rnorm(10000),nclas=100,freq=FALSE,
+  main="Symmetric Distribution", xlab="rnorm")
> hist.flat <- hist(runif(10000),nclas=100,freq=FALSE,
+  main="Symmetric Flat Distribution", xlab="runif")
> hist.skr <- hist(rgamma(10000,shape=2,scale=1),freq=FALSE, nclas=100,
+  main="Skewed to Right", xlab="rgamma")
> hist.skl <- hist(rbeta(10000,8,2),nclas=100,freq=FALSE,
+  main="Skewed to Left", xlab="rbeta")
```



隨機抽樣 (Random Sampling)

- The concepts of randomness and probability are central to statistics.

```
> sample(x, size, replace = FALSE, prob = NULL)
```

- sampling without replacement

```
> sample(1:40, 5)
[1] 12 38  2  3  7
```

- sampling with replacement

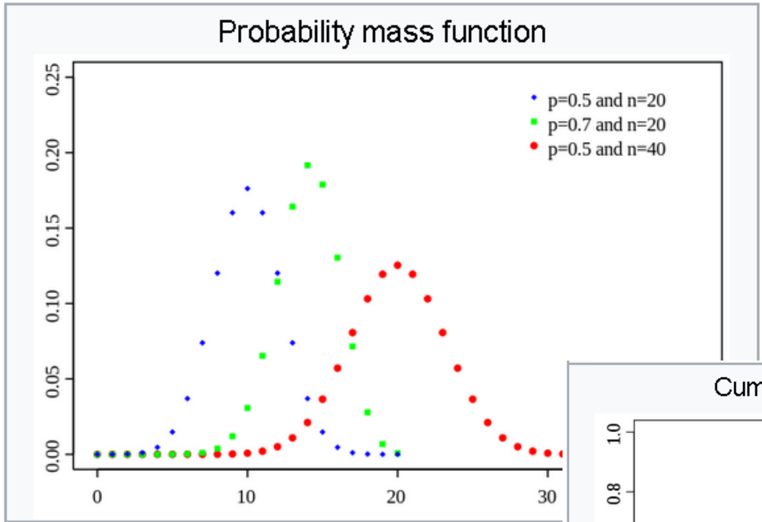
```
> sample(1:40, 5, replace=TRUE)
[1] 35  4  4 16 22
```

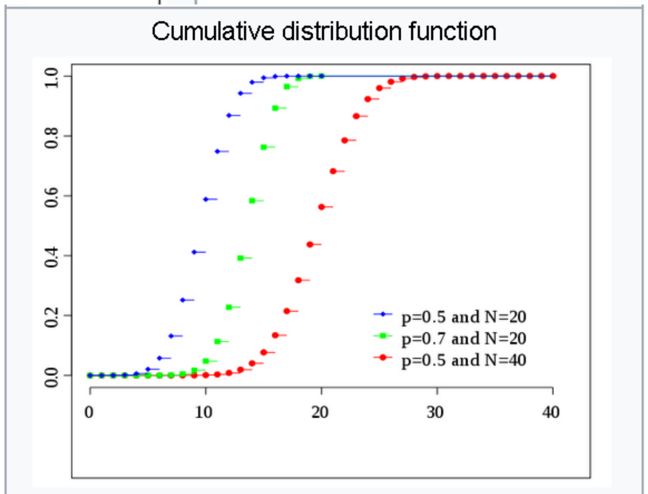
- Simulate 10 coin tosses (fair coin-tossing)

```
> sample(c("H", "T"), 10, replace=T)
 [1] "T" "T" "T" "H" "H" "H" "T" "H" "T" "H"
> sample(c("succ", "fail"), 10, replace=T, prob=c(0.9, 0.1))
 [1] "succ" "succ" "succ" "fail" "fail" "fail" "succ" "succ" "succ" "succ"
```

二項式分佈 (Binomial)

- X~B(n, p)表示n次伯努利試驗中(size)，成功結果出現的次數。
- 例: 擲一枚公正銅板十次，那麼擲得正面次數就服從n = 10、p = 1/2 的二項分布。
- `dbinom(x, size, prob)` # 機率公式值 P(X=x)
- `pbinom(q, size, prob)` # 累加至q的機率值 P(X <= q)
- `qbinom(p, size, prob)` # 已知累加機率值，對應的機率點。
- `rbinom(n, size, prob)` # 隨機樣本數=n的二項隨機變數值。





Notation	$B(n, p)$
Parameters	$n \in \mathbf{N}_0$ — number of trials $p \in [0,1]$ — success probability in each trial
Support	$k \in \{0, \ldots, n\}$ — number of successes
pmf	$\binom{n}{k} p^k (1-p)^{n-k}$
CDF	$I_{1-p}(n-k, 1+k)$
Mean	np
Median	$\lfloor np \rfloor$ or $\lceil np \rceil$
Mode	$\lfloor (n+1)p \rfloor$ or $\lceil (n+1)p \rceil - 1$
Variance	$np(1-p)$
Skewness	$\frac{1-2p}{\sqrt{np(1-p)}}$
Ex. kurtosis	$\frac{1-6p(1-p)}{np(1-p)}$
Entropy	$\frac{1}{2}\log_2\left(2\pi e\, np(1-p)\right) + O\left(\frac{1}{n}\right)$ in shannons. For nats, use the natural log in the log.
MGF	$(1-p+pe^t)^n$
CF	$(1-p+pe^{it})^n$
PGF	$G(z) = [(1-p)+pz]^n.$
Fisher information	$g_n(p) = \frac{n}{p(1-p)}$ (for fixed n)

https://en.wikipedia.org/wiki/Binomial_distribution

二項式分佈

X~B(10, 0.8)

- 利用二項分配理論公式，計算機率公式值 P(X=3)。

```
> factorial(10)/(factorial(3)*factorial(7))*0.8^3*0.2^7
[1] 0.000786432
```

- 利用R函數，計算機率值 P(X=3)。

```
> dbinom(3, 10, 0.8)
[1] 0.000786432
```

- 計算P(X<= 3)- P(X<= 2) ，並和P(X=3)相比較。

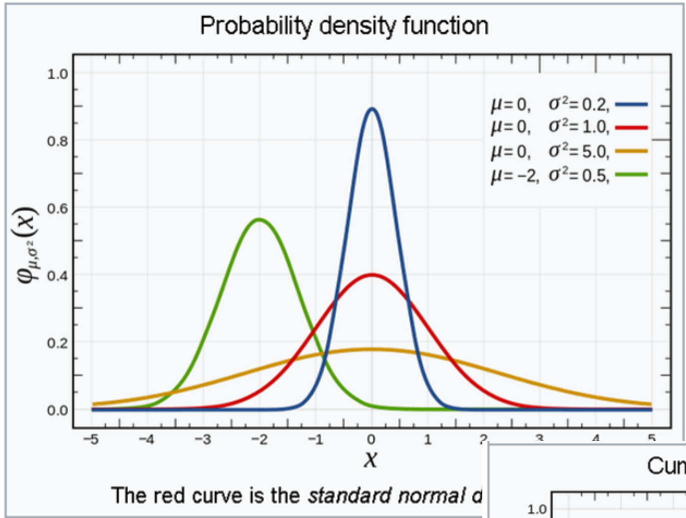
```
> pbinom(3, 10, 0.8)- pbinom(2, 10, 0.8)
[1] 0.000786432
```

- 已知累加機率值為0.1208，求對應的分位數。

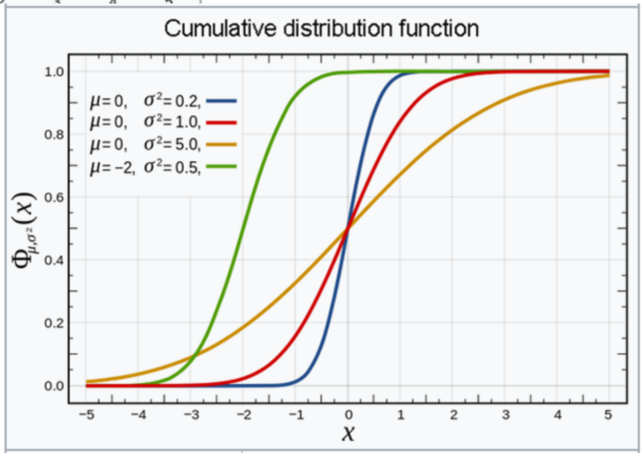
```
> qbinom(0.1208, 10, 0.8)
[1] 6
> pbinom(6, 10, 0.8)
[1] 0.1208739
```

常態分佈 (Normal Distribution)

- `dnorm(x, mean, sd)`#機率密度函數值 f(x)
- `pnorm(q, mean, sd)`#累加機率值P(X<= x)
- `qnorm(p, mean, sd)`#累加機率值p 對應的分位數
- `rnorm(n, mean, sd)`#常態隨機樣本



The red curve is the *standard normal d*



Notation	$\mathcal{N}(\mu,\, \sigma^2)$
Parameters	$\mu \in \mathbf{R}$ — mean (location) $\sigma^2 > 0$ — variance (squared scale)
Support	$x \in \mathbf{R}$
PDF	$\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
CDF	$\frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$
Quantile	$\mu + \sigma\sqrt{2}\,\operatorname{erf}^{-1}(2F-1)$
Mean	μ
Median	μ
Mode	μ
Variance	σ^2
Skewness	0
Ex. kurtosis	0
Entropy	$\frac{1}{2}\ln(2\sigma^2\pi\, e)$
MGF	$\exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}$
CF	$\exp\{i\mu t - \frac{1}{2}\sigma^2 t^2\}$
Fisher information	$\begin{pmatrix} 1/\sigma^2 & 0 \\ 0 & 1/(2\sigma^4) \end{pmatrix}$

https://en.wikipedia.org/wiki/Normal_distribution

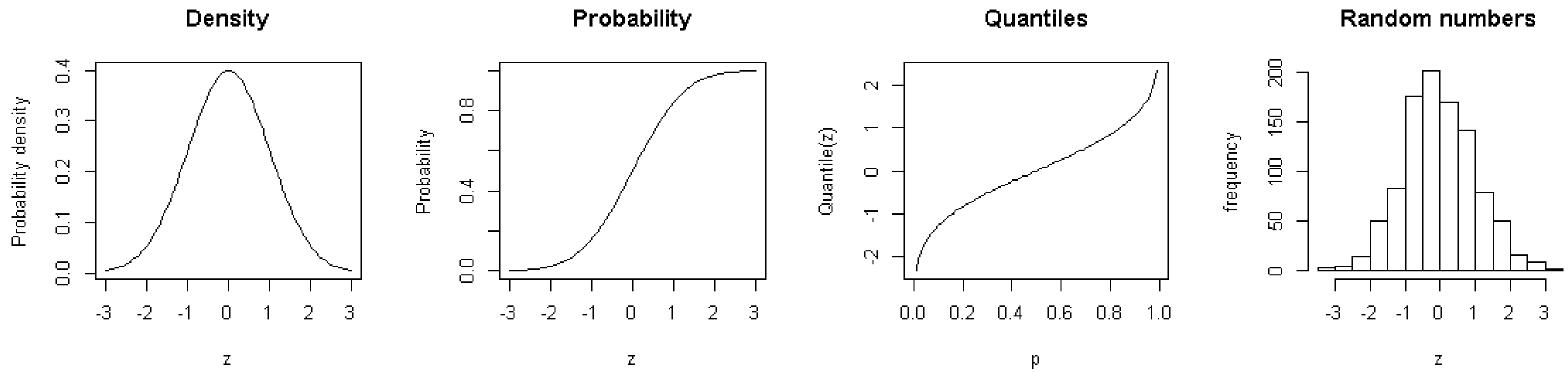
常態分佈 (Normal Distribution)

Z ~N(0, 1)

```
> dnorm(0)
[1] 0.3989423
> pnorm(-1)
[1] 0.1586553
> qnorm(0.975)
[1] 1.959964
```

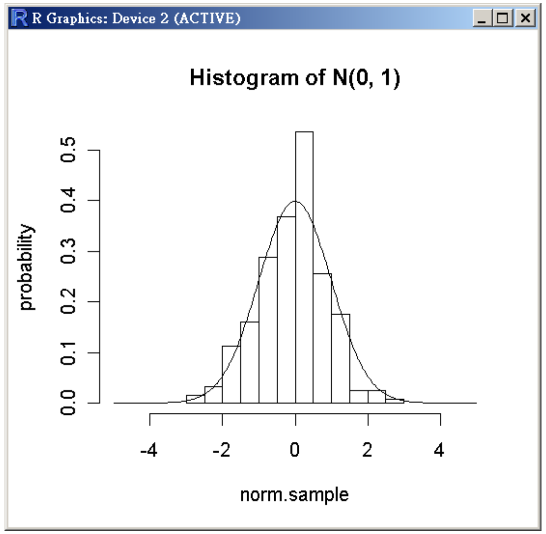
```
> dnorm(10, 10, 2) # X~N(10, 4)
[1] 0.1994711
> pnorm(1.96, 10, 2)
[1] 2.909907e-05
> qnorm(0.975, 10, 2)
[1] 13.91993
> rnorm(5, 10, 2)
[1]  9.043357 11.721717  7.763277  9.563463 10.072386
> pnorm(15, 10, 2) - pnorm(8, 10, 2)  # P(8<=X<=15)
[1] 0.8351351
```

```
> par(mfrow=c(1,4))
> curve(dnorm, -3, 3, xlab="z", ylab="Probability density", main="Density")
> curve(pnorm, -3, 3, xlab="z", ylab="Probability", main="Probability")
> curve(qnorm, 0, 1, xlab="p", ylab="Quantile(z)", main="Quantiles")
> hist(rnorm(1000), xlab="z", ylab="frequency", main="Random numbers")
```



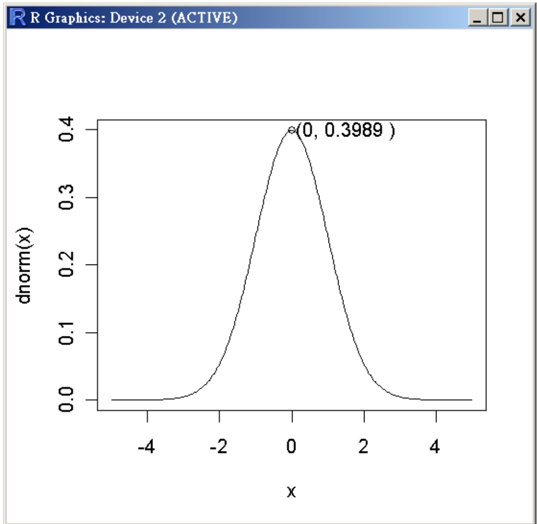
常態分佈

```
> norm.sample <- rnorm(250)
> summary(norm.sample)
> hist(norm.sample, xlim=c(-5, 5), ylab="probability",
+  main="Histogram of N(0, 1)", prob=T)
> x <- seq(from=-5, to=5, length=300)
> lines(x, dnorm(x))
```



標出最頂點的座標

```
> x <- seq(from=-5, to=5, length=300)
> plot(x, dnorm(x), type="l")
> points(0, dnorm(0))
> height <- round(dnorm(0), 4); height
> text(1.5, height, paste("(0,", height, ")"))
```



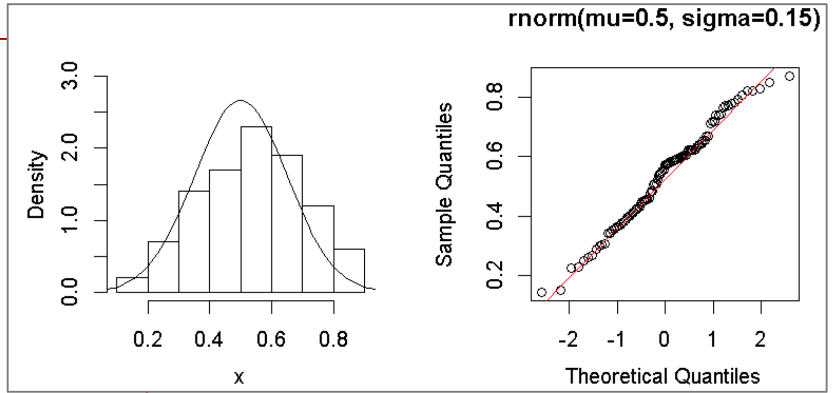
Quantile-Quantile Plots : Testing for Normality Graphically

- The quantile-quantile (Q-Q) plot is used to determine if two data sets come from populations with a common density.
- Q-Q plots are sometimes called probability plots, especially when data are examined against a theoretical density.

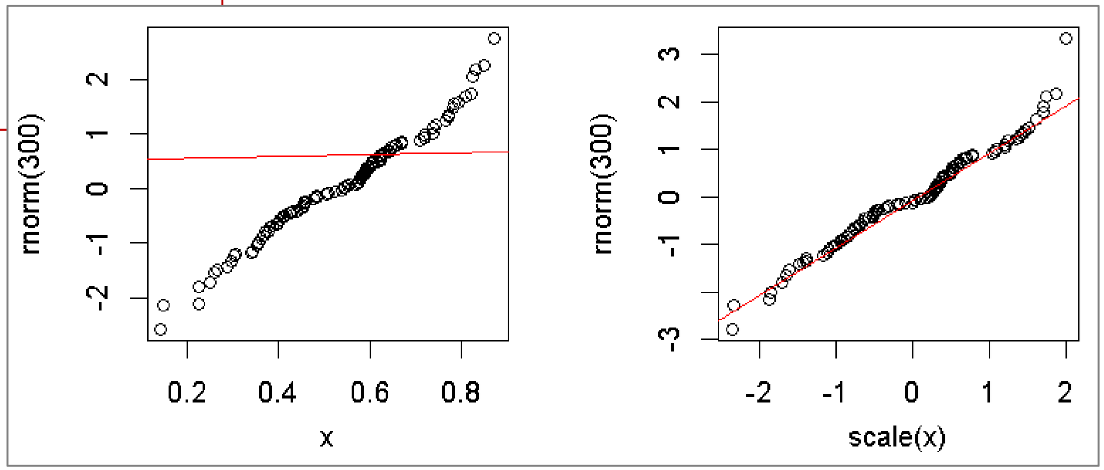
- **`qqnorm()`**: produces a normal QQ plot of the values in sample
- **`qqline()`**: adds a line which passes through the first and third quartiles.
  - Use the diagonal line would not make sense because the first axis is scaled in terms of the theoretical quantiles of a N (0,1) distribution.
  - Using the first and third quartiles to set the line gives a robust approach for estimating the parameters of the normal distribution, when compared with using the empirical mean and variance, say.
  - Departures from the line (except in the tails) are indicative of a lack of normality.
- **`qqplot()`**: qqplot produces a QQ plot of two datasets.

qqnorm, qqline, qqplot

```
> par(mfrow = c(1, 2))
> set.seed(12345);
> n <- 100; mu <- 0.5; sigma <- 0.15
> x <- rnorm(n, mu, sigma)
> hist(x, freq=FALSE, ylim=c(0, 3), main="")
> y <- seq(0, 1, length = n)
> lines(y, dnorm(y, mu, sigma), type = 'l')
> qqnorm(x, main = "rnorm(mu=0.5, sigma=0.15)");
> qqline(x)
```



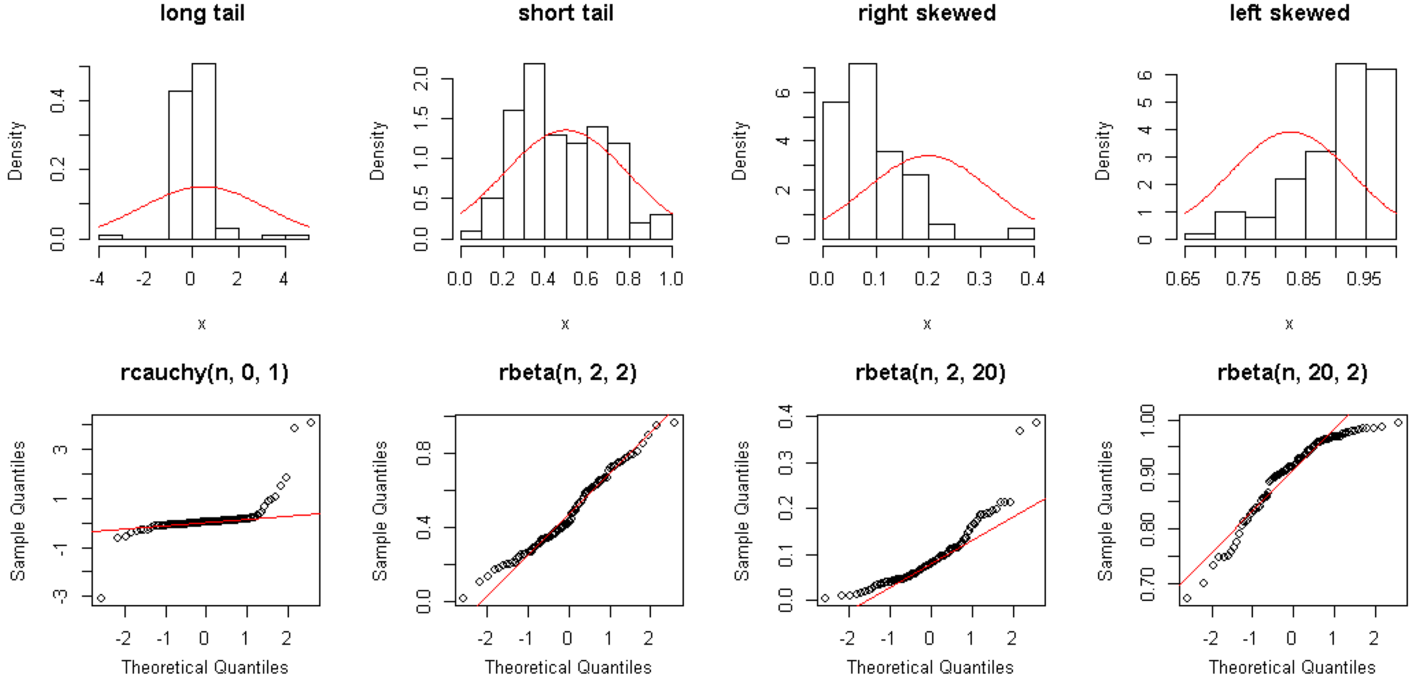

```
> qqplot(x, rnorm(300))
> qqline(x, col = 2)
> qqplot(scale(x), rnorm(300))
> qqline(scale(x), col = 2)
```



課堂練習

```
par(mfcol=c(2,4))
n <- 100
x <- rcauchy(n, 0, 0.1) # a long tail density
hist(x, freq=FALSE, main="long tail")
curve(dnorm(x, mean=mean(x), sd=sd(x)), add=TRUE, col="red")
qqnorm(x, main = "rcauchy(n, 0, 1)");
qqline(x, col="red")
# different shapes of distributions
x <- rbeta(n, 2, 2) # a short tail density
x <- rbeta(n, 2, 20) # a right skewed density
x <- rbeta(n, 20, 2) # a left skewed density
```

Try
- Normal
- Symmetric, Non-Normal, Short-Tailed
- Symmetric, Non-Normal, Long-Tailed
- Symmetric and Bimodal
- Bimodal Mixture of 2 Normals
- Skewed (Non-Symmetric) Right
- Skewed (Non-Symmetric) Left
- Symmetric with Outlier



實作 Quantile-Quantile Plots

$$(X_1, X_2, \cdots, X_n)$$

1. 計算樣本平均數及樣本變異數。

$$\bar{X} = \frac{\sum X_i}{n} \qquad S^2 = \frac{1}{n-1}\sum(X_i - \bar{X})^2$$

2. 將隨機樣本標準化並排序。

$$d_{(1)}, d_{(2)}, \cdots, d_{(n)}$$

$$d_{(i)} = \frac{X_i - \bar{X}}{S}$$

3. 查出 n個標準常態值: (將標準態分配，區分成n+1區塊，最左及最右區塊的機率分別為 1/2n, 中間的 n-1區塊, 機率分別為1/n)。

$$q_{(1)} = z_{\frac{1}{2n}}, q_{(2)} = z_{\frac{3}{2n}}, \cdots, q_{(n)} = z_{\frac{2n-1}{2n}}$$

$$q_{(i)} = z_{\frac{2i-1}{2n}}$$

$$P(Z < q_{(i)}) = \frac{2i-1}{2n}$$

4. 畫散佈圖: x軸: 排序的標準化樣本，y軸: 標準常態值。

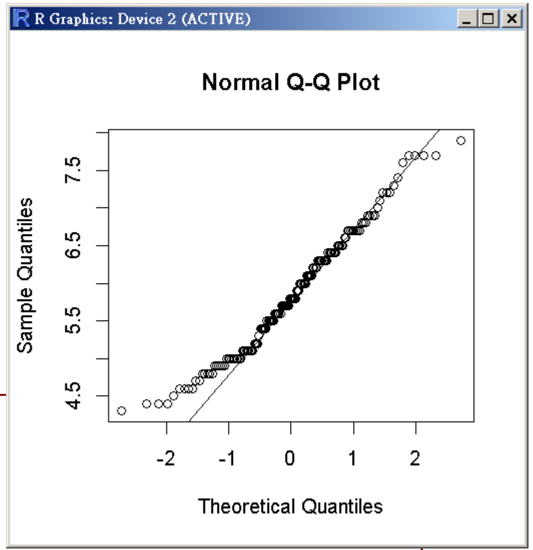
$$(d_{(i)}, q_{(i)})$$

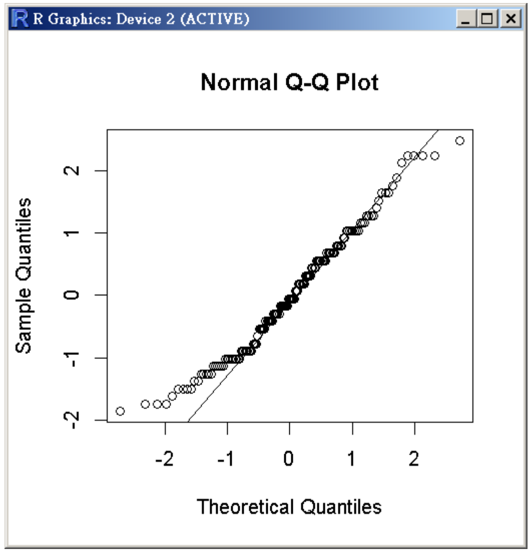
5. 加入一條由(q(i), q(i))產生的標準常態直線。
(或加入一條通過第25%及第75%quantiles的直線)

$$(q_{(i)}, q_{(i)})$$

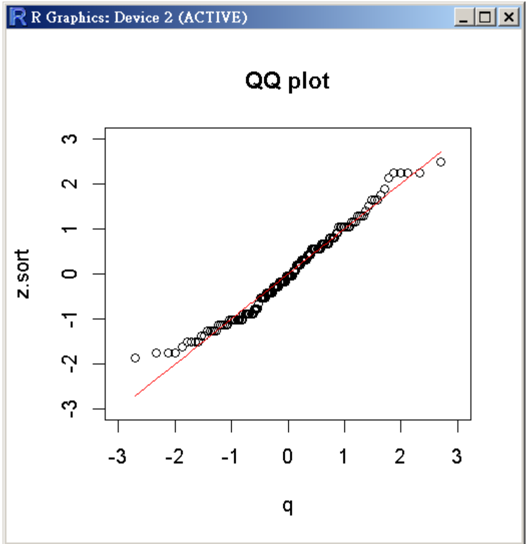
課堂練習

```
> qqnorm(iris[,1])
> qqline(iris[,1])

> qqnorm(scale(iris[,1]))
> qqline(scale(iris[,1]))

> my.qqplot(iris[,1])
```






```
my.qqplot <- function(x){
   x.mean <- mean(x)
   x.var <- var(x)
   n <- length(x)

   z <- (x-x.mean)/sqrt(x.var)
   z.mean <- mean(z)
   z.var <- var(z)
   z.sort <- sort(z)

   k <- 1:n
   p <- (k-0.5)/n
   q <- qnorm(p)

   plot(q, z.sort, xlim=c(-3, 3), ylim=c(-3, 3))
   title("QQ plot")
   lines(q, q, col=2)
}
```



中央極限定理 (Central Limit Theorem)

- 由一具有平均數μ，標準差σ的母體中抽取樣本大小為n的簡單隨機樣本，當樣本大小n夠大時，**樣本平均數的抽樣分配**會近似於常態分配。

$X_1, X_2, X_3, \cdots$ be a set of n independent and identically distributed random variables having finite values of mean μ and variance $\sigma^2 > 0$.

$$S_n = X_1 + \cdots + X_n$$

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} \to N(0,1) \quad \text{as} \quad n \to \infty$$

$$Z_n = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$$

$$E(\bar{X}) = \mu_{\bar{X}} = \mu$$

$$Var(\bar{X}) = \sigma_{\bar{X}}^2 = \frac{\sigma^2}{n}$$

Data	x1	x2	x3	x4	•••	xp
subject01	-0.48	-0.42	0.87	0.92		-0.18
subject02	-0.39	-0.58	1.08	1.21		-0.33
subject03	0.87	0.25	-0.17	0.18		-0.44
subject04	1.57	1.03	1.22	0.31		-0.49
subject05	-1.15	-0.86	1.21	1.62		0.16
subject06	0.04	-0.12	0.31	0.16		-0.06
subject07	2.95	0.45	-0.40	-0.66		-0.38
subject08	-1.22	-0.74	1.34	1.50		0.29
subject09	-0.73	-1.06	-0.79	-0.02		0.44
subject10	-0.58	-0.40	0.13	0.58		0.02
subject11	-0.50	-0.42	0.66	1.05		0.06
subject12	-0.86	-0.29	0.42	0.46		0.10
subject13	-0.16	0.29	0.17	-0.28		-0.55
subject14	-0.36	-0.03	-0.03	-0.08		-0.25
subject15	-0.72	-0.85	0.54	1.04		0.24
subject16	-0.78	-0.52	0.26	0.20		0.48
subject17	0.60	-0.55	0.41	0.45		-0.66
⋮						
subject n	-2.29	-0.64	0.77	1.60		0.55
mean	0.07	-0.04	0.44	0.31	•••	-0.21

- 在一般的統計實務上，大部分的應用中均假設當樣本大小為30(含)以上時，$\bar{X}_n$的抽樣分配即近似於常態分配。
- 當母體為常態分配時，不論樣本大小，樣本平均數的抽樣分配仍為常態分配。

範例: 應用CLT算機率

- 於某考試中，考生之通過標準機率為0.7，以隨機變數表示考生之通過與否 (X=1表示通過) (X=0表示不通過)，其機率分配為 P(X=1)=0.7, P(X=0)=0.3。
  1. 計算母體平均數及變異數。
  2. 假如有210名考生，計算「平均通過人數」的平均數及變異數。
  3. 計算通過人數> 126的機率。

X ~B(n, p) 二項式分布
http://shorturl.at/iLUV5

1.

$$\mu = E(X) = p = 0.7$$

$$\sigma^2 = Var(X) = p(1-p) = 0.21$$

2.

$$X_1, X_2, \cdots, X_{210}:$$

$$X_i = 1 : \text{success}$$

$$X_i = 0 : \text{fail}$$

$$\bar{X}_{210} = \frac{X_1 + \cdots + X_{210}}{210}$$

$$\mu_{\bar{X}} = \mu = 0.7$$

$$\sigma^2_{\bar{X}} = \frac{\sigma^2}{210} = 0.001$$

3.

$$P(X_1 + X_2 + \cdots + X_{210} > 126)$$

$$= P(\bar{X} > \frac{126}{210})$$

$$= P(\bar{X} > 0.6)$$

$$= P(Z > \frac{0.6 - 0.7}{\sqrt{0.001}})$$

$$= P(Z > -3.16228)$$

$$= 0.99922$$

應用CLT

課堂練習

```
> z <- (126/210 -0.7)/sqrt(0.001) # 通過人數>126的機率
> z
[1] -3.162278
> 1 - pnorm(z)
[1] 0.9992173
```

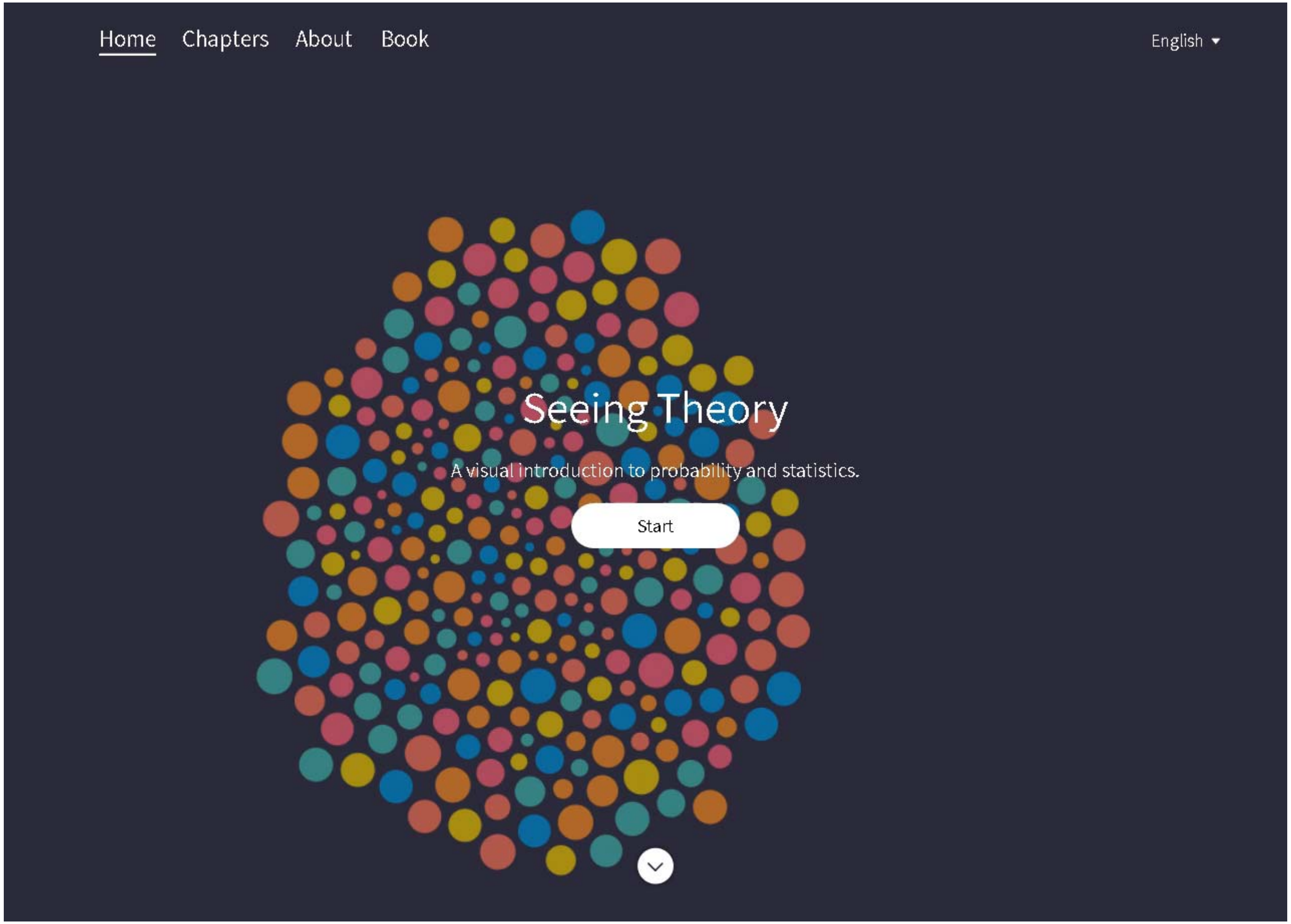
寫一「通過人數大於某數的機率」之副程式

- n: 考生總數(n=210)
- X: 通過考生之人數, X~B(210, 0.7)

```
> pass.prob <- function(x, n, mu, sigma2, digit = m){
    xbar <- x/n
    z <- (xbar-mu)/sqrt(sigma2)
    zvalue <- round(z, digit)
    right.prob <- round(1-pnorm(z), digit)
    list(zvalue = zvalue, prob = right.prob)
  }

> pass.prob(126, 210, 0.7, 0.001, 4)
$zvalue
[1] -3.1623

$prob
[1] 0.9992
```

中央極限定理:
樣本平均之抽樣分佈

https://students.brown.edu/seeing-theory/



驗証中央極限定理

1. 先做隨機樣本的取樣。

$X \sim D(\cdot)$ $X_1, X_2, \cdots, X_{m_0} \sim D(\cdot)$ $m = m_0$

2. 計算樣本平均。

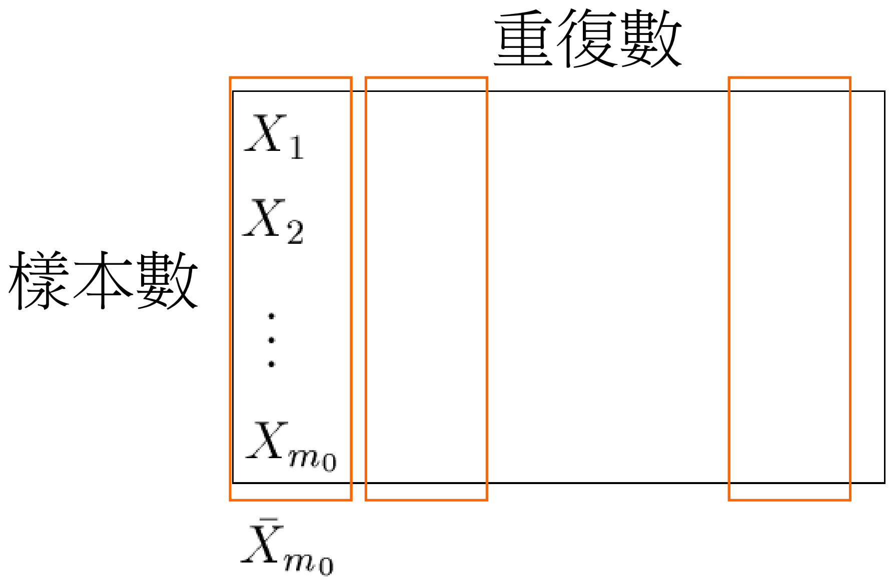
$$\bar{X}_{m_0} = \frac{1}{m_0}(X_1 + X_2 + \cdots + X_{m_0})$$

3. 重復上述動作數百或數仟次，得到抽樣平均的分佈。
4. 描繪出抽樣平均之抽樣分配直方圖。
5. 畫出相對應的qqplot。
6. 再做各種不同樣本數(m_0=1, 5, 15, 30,...)的抽樣計算。

範例: Uniform Distribution

$$X_1, X_2, \cdots \sim U(5, 80)$$

$$\bar{X}_n = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$$



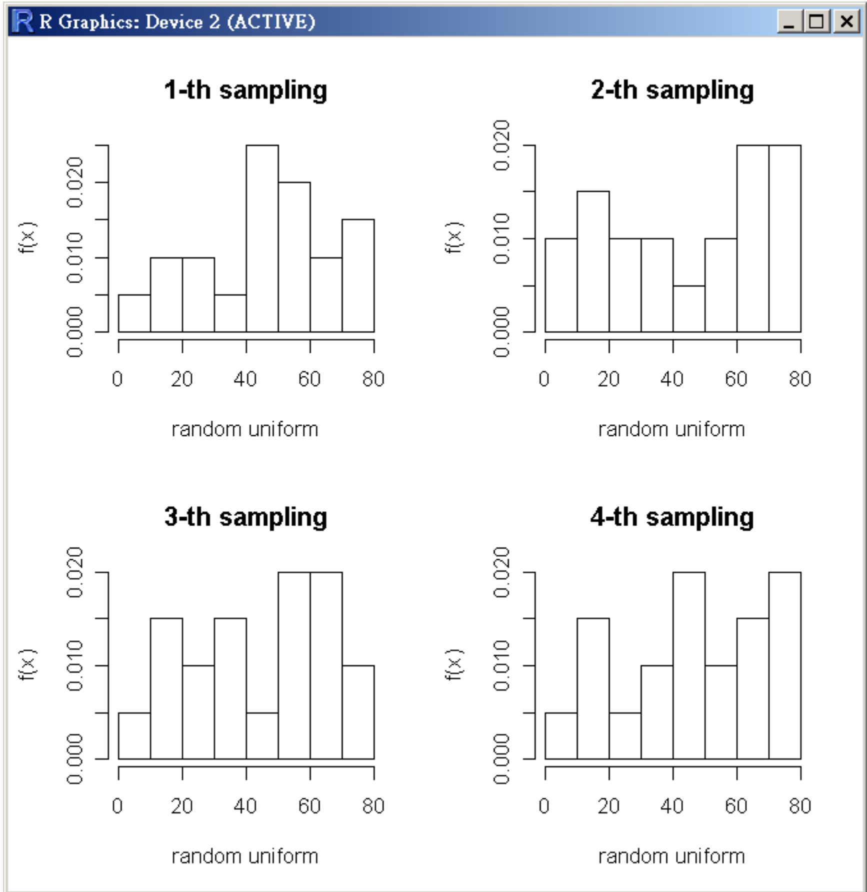

```
umin <- 5
umax <- 80
n.sample <- 20
n.repeated <- 500

RandomSample <- matrix(0, n.sample, n.repeated)
for(i in 1:n.repeated){
   rnumber <- runif(n.sample, umin, umax)
   RandomSample[,i] <- as.matrix(rnumber)

}
dim(RandomSample)
```

抽樣樣本之直方圖

```
par(mfrow=c(2,2))
for(i in 1:4){
  title <- paste(i,"-th sampling", sep="")
  hist(RandomSample[,i], ylab="f(x)", xlab="random uniform", pro=T, main=title)
}
```

$$X_1, X_2, \cdots \sim U(5, 80)$$

$$\bar{X}_n = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$$



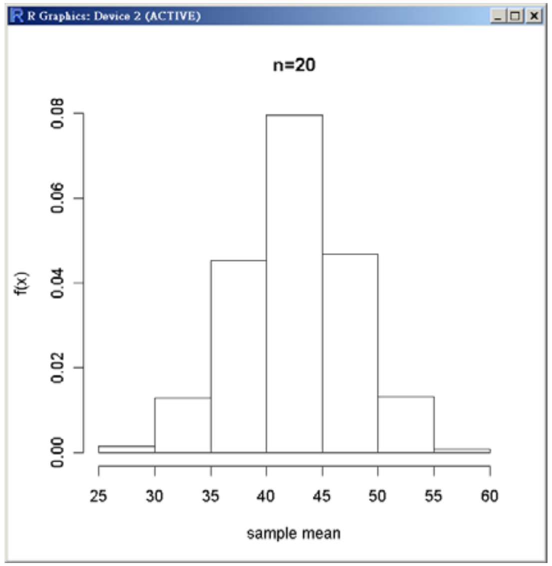
抽樣樣本平均之直方圖&QQplot

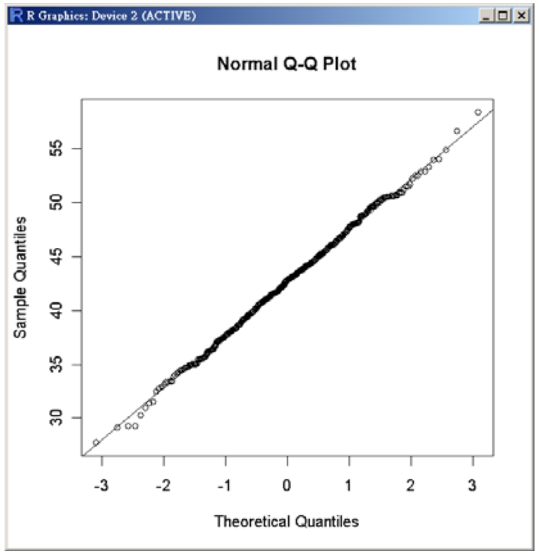
```
> SampleMean <- apply(RandomSample, 2, mean)
> hist(SampleMean, ylab="f(x)", xlab="sample mean", pro=T, main="n=20")
```

```
myfun <- function(x){
  mu <- (umin + umax)/2
  s2 <- (umax - umin)^2/12
  z <- (mean(x) - mu)/sqrt(s2/n)
  z
}
```

$$X_1, X_2, \cdots \sim U(5, 80)$$

$$\bar{X}_n = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$$





Notation	$\mathcal{U}(a,b)$ or $\mathrm{unif}(a,b)$
Parameters	$-\infty < a < b < \infty$
Support	$x \in [a,b]$
PDF	$\begin{cases} \frac{1}{b-a} & \text{for } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$
CDF	$\begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } x \in [a,b] \\ 1 & \text{for } x > b \end{cases}$
Mean	$\frac{1}{2}(a+b)$
Median	$\frac{1}{2}(a+b)$
Mode	any value in (a,b)
Variance	$\frac{1}{12}(b-a)^2$

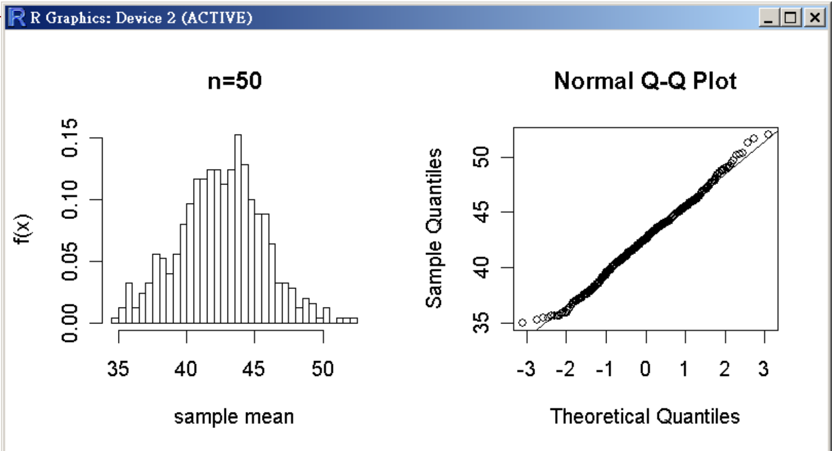
https://en.wikipedia.org/wiki/Continuous_uniform_distribution

```
> qqnorm(SampleMean)
> qqline(SampleMean)
```

重複不同的樣本數

```
CLT.unif <- function(umin, umax, n.sample, n.repeated){
    RandomSample <- matrix(0, n.sample, n.repeated)
    for(i in 1:n.repeated){
       rnumber <- runif(n.sample, umin, umax)
       RandomSample[,i] <- as.matrix(rnumber)

    }
    SampleMean <- apply(RandomSample, 2, mean)
    par(mfrow=c(1,2))
    title <- paste("n=",n.sample, sep="")
    hist(SampleMean, breaks=30, ylab="f(x)", xlab="sample mean", freq=F,
main=title)
    qqnorm(SampleMean)
    qqline(SampleMean)
}
```

```
myfun <- function(x){
  mu <- (umin + umax)/2
  s2 <- (umax - umin)^2/12
  z <- (mean(x) - mu)/sqrt(s2/n)
  z
}
```

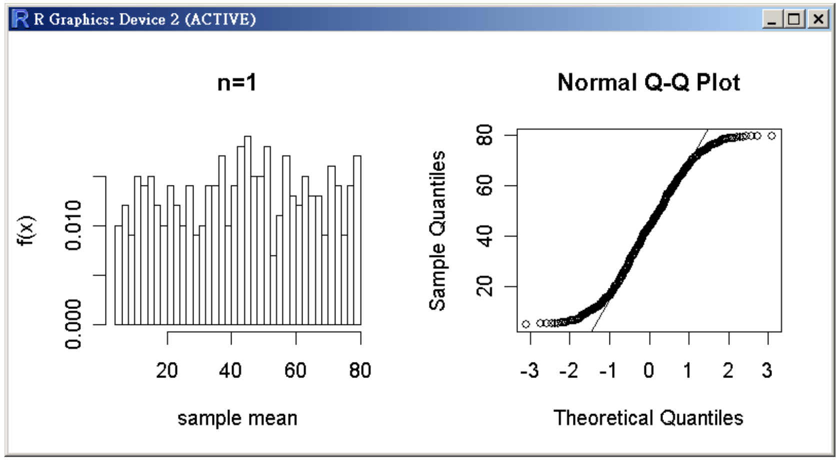



```
CLT.unif(5, 80, 50, 500)
```

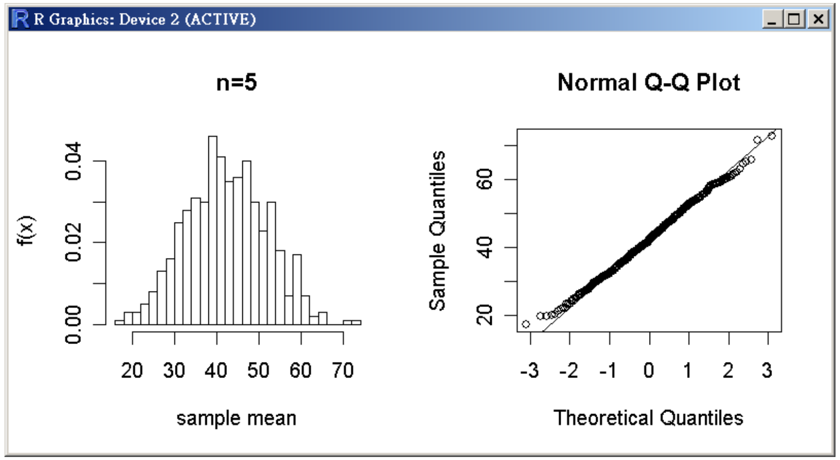
當樣本數*n*愈大時，從樣本平均數的抽樣分配可以得到「中央極限定理」的主要結論。

CLT.unif(umin, umax, n.sample, n.repeated

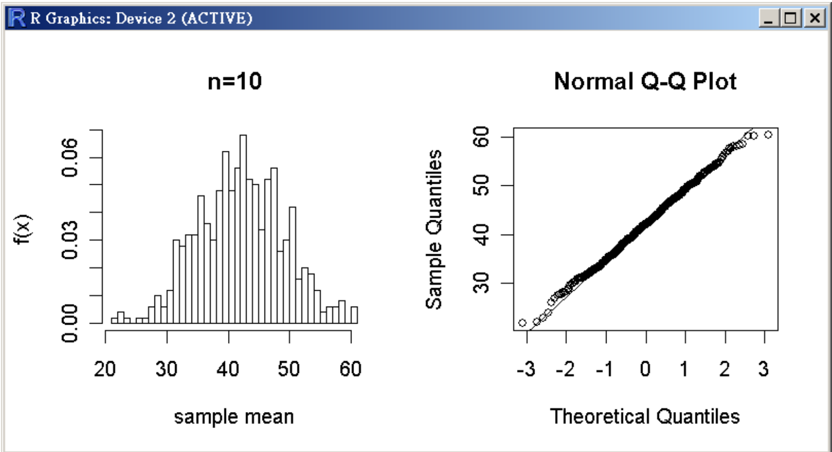
CLT.unif(5, 80, 1, 500)



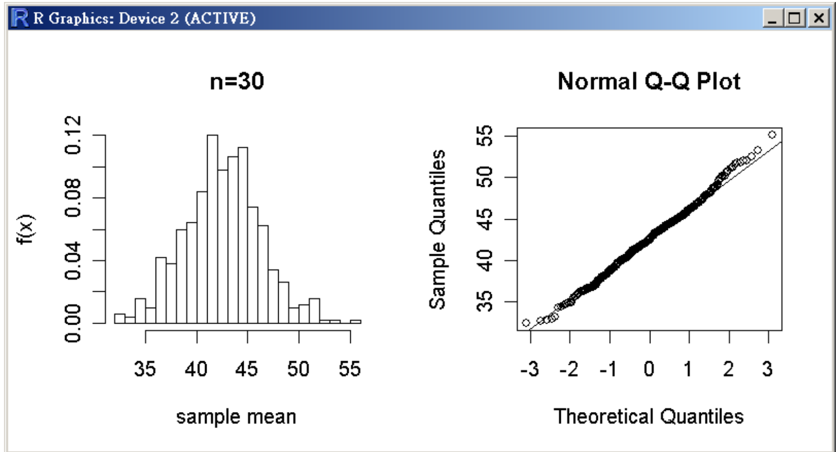
CLT.unif(5, 80, 5, 500)



CLT.unif(5, 80, 10, 500)



CLT.unif(5, 80, 30, 500)



用R程式模擬算機率: 我們要生女兒

一對夫婦計劃生孩子生到有女兒才停，或生了三個就停止。他們會擁有女兒的機率是多少？

- **第I 步：機率模型**
  - 每一個孩子是女孩的機率是0.49 ，是男孩的機率是0.51。各個孩子的性別是互相獨立的。
- **第2 步：分配隨機數字。**
  - 用兩個數字模擬一個孩子的性別: 00, 01, 02, ..., 48 ＝ 女孩; 49, 50, 51, ..., 99 ＝ 男孩
- **第3 步：模擬生孩子策略**
  - 從表A當中讀取一對一對的數字，直到這對夫婦有了女兒，或已有三個孩子。

6905	16	48	17	8717	40	9517	845340	648987	20
男女	女	女	女	男女	女	男女	男男女	男男男	女
+	+	+	+	+	+	+	+	−	+

- 10次重複中，有9次生女孩。會得到女孩的機率的估計是9/10=0.9。
- 如果機率模型正確的話，用數學計算會有女孩的真正機率是0.867。(我們的模擬答案相當接近了。除非這對夫婦運氣很不好，他們應該可以成功擁有一個女兒。)

用R程式模擬算機率: 我們要生女兒

```r
girl.born <- function(n, show.id = F){

  girl.count <- 0
  for (i in 1:n) {
    if (show.id) cat(i,": ")
    child.count <- 0
    repeat {
        rn <- sample(0:99, 1, replace=T)
        if (show.id) cat(paste0("(", rn, ")"))
        is.girl <- ifelse(rn <= 48, TRUE, FALSE)
        child.count <- child.count + 1
        if (is.girl){
          girl.count <- girl.count + 1
          if (show.id) cat("女+")
          break
        } else if (child.count == 3) {
          if (show.id) cat("男")
          break
        } else{
          if (show.id) cat("男")
        }
    }
    if (show.id) cat("\n")
  }
  p <- girl.count / n
  p

}
```

```r
> girl.p <- 0.49 + 0.51*0.49 + 0.51^2*0.49
> girl.p
[1] 0.867349
>
> girl.born(n=10, show.id = T)
1 : (73)男(18)女+
2 : (23)女+
3 : (53)男(74)男(64)男
4 : (95)男(20)女+
5 : (63)男(16)女+
6 : (48)女+
7 : (67)男(51)男(44)女+
8 : (74)男(99)男(25)女+
9 : (47)女+
10 : (81)男(41)女+
[1] 0.9
> girl.born(n=10000)
[1] 0.8674
```