Regression Analysis (I)

Kutner's Applied Linear Statistical Models (5/E)

Chapter 5: Matrix Approach to Simple Linear Regression Analysis

Thursday 09:10-12:00, 商館 260205

Han-Ming Wu

Department of Statistics, National Chengchi University

http://www.hmwu.idv.tw

Overview

1. The matrix approach is practically a necessity in ____________ regression analysis, since it permits extensive systems of equations and large arrays of data to be denoted compactly and operated upon efficiently.
2. This chapter gives a brief introduction to amatrix algebra.
3. Then we apply matrix methods to the simple linear regression model.

5.1 Matrices

Definition of Matrix

1. A matrix is a ______________ array of elements arranged in rows and columns.
2. A matrix with __________ and _____________ will be represented either in full:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1c} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2c} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{ic} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rj} & \cdots & a_{rc} \end{bmatrix}$$

or in abbreviated form:

$$\mathbf{A} = \underline{\hspace{2cm}}, \ i = 1, \cdots, r; j = 1, \cdots, c$$

or simply by a boldface symbol, such as **A**.

Square Matrix

1. A matrix is said to be square if the number of rows ________ the number of columns.

Vector

1. A matrix containing only one column is called a __________ vector or simply a vector.

$$\mathbf{C} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix}$$

the vector **C** is a ______________.

2. A matrix containing only one row is called a ____________: e.g., $\mathbf{B}' = [15\ 25\ 50]$. We use the prime symbol (____________) for row vectors. Note that the row vector **B**' is a ________ matrix.

Transpose

1. The transpose of a matrix **A** is another matrix, denoted by ______', that is obtained by interchanging corresponding columns and rows of the matrix **A**.

$$\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 7 & 10 \\ 3 & 4 \end{bmatrix}$$

then the transpose **A**' is:

$$\mathbf{A}' = \underline{\hspace{4cm}}$$

2. The transpose of a column vector is a row vector, and vice versa. This is the reason why we used the symbol **B**' earlier to identify a row vector, since it may be thought of as the transpose of a column vector **B**. In general, we have:

$$\mathbf{A} = [a_{ij}], \quad \underline{\qquad\qquad\qquad}$$

Equality of Matrices

1. Two matrices **A** and **B** are said to be equal if they have the same dimension and if all corresponding ____________________.

Regression Examples

1. In regression analysis, one basic matrix is the vector **Y**, consisting of the n observations on response variable

$$\mathbf{Y} =$$

$$\underline{\qquad\qquad\qquad}$$

2. Another basic matrix in regression analysis is the **X** matrix, which is defined as follows for simple linear regression analysis:

$$\mathbf{X} =$$

$$\underline{\qquad\qquad\qquad}$$

The matrix **X** consists of a column of 1s and a column containing the n observations on the predictor variable X. The **X** matrix is often referred to as the design matrix.

5.2 Matrix Addition and Subtraction

1. Adding or subtracting two matrices requires that they have the same dimension. The sum, or difference, of two matrices is another matrix whose elements each consist of the sum, or difference, of the corresponding elements of the two matrices.

2.

$$\text{if } \mathbf{A}_{r\times c} = [a_{ij}], \quad \mathbf{B}_{r\times c} = [b_{ij}], \text{ then } \mathbf{A} \pm \mathbf{B} = \underline{\hspace{3cm}}$$

3. The regression model: $Y_i = E(Y_i) + \varepsilon_i,\ i = 1, \cdots, n$ can be written in matrix notation:

$$\underline{\hspace{4cm}}$$

4. The observations vector $\mathbf{Y}$ equals the sum of two vectors, a vector containing the expected values and another containing the error terms.

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} E(Y_1) \\ E(Y_2) \\ \vdots \\ E(Y_n) \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix} = \begin{bmatrix} E(Y_1) + \varepsilon_1 \\ E(Y_2) + \varepsilon_2 \\ \vdots \\ E(Y_n) + \varepsilon_n \end{bmatrix}$$

5.3 Matrix Multiplication

Multiplication of a Matrix by a Scalar

1. A scalar is an ordinary number or a symbol representing a number. In multiplication of a matrix by a scalar, every element of the matrix is multiplied by the scalar.

2. If $\mathbf{A} = [a_{ij}]$ and k is the scalar, then

$$k\mathbf{A} = \mathbf{A}k = \underline{\hspace{2cm}}$$

Multiplication of a Matrix by a Matrix

1. In general, the product $\mathbf{AB}$ is defined only when the number of columns in $\mathbf{A}$ equals the number of rows in $\mathbf{B}$ so that there will be corresponding terms in the __________________.

2. Note that the dimension of the product $\mathbf{AB}$ is given by the number of rows in $\mathbf{A}$ and the number of columns in $\mathbf{B}$. Note also that in the second case the product $\mathbf{BA}$ would not be defined since the number of columns in $\mathbf{B}$ is not equal to the number of rows in $\mathbf{A}$.

3. In general, if $\mathbf{A} = [a_{ik}]$ has dimension $r \times c$ and $\mathbf{B} = [b_{kj}]$ has dimension $c \times s$, the product $\mathbf{AB}$ is a matrix of dimension $r \times s$ whose element in the ith row and jth column is:

$$\mathbf{AB} = \underline{\hspace{4cm}}$$

Regression Examples

1. A product frequently needed is $\mathbf{Y'Y}$, where $\mathbf{Y}$ is the vector of observations on the response variable

$$\mathbf{Y'Y} = [Y_1\ Y_2\ \cdots\ Y_n]\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \underline{\hspace{5cm}} = \underline{\hspace{3cm}}$$

2. $\mathbf{X'X}$ is a 2×2 matrix:

$$\mathbf{X'X} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ X_1 & X_2 & \cdots & X_n \end{bmatrix}\begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_n \end{bmatrix} = \underline{\hspace{5cm}}$$

3. $\mathbf{X'Y}$ is a 2×1 matrix:

$$\mathbf{X'Y} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ X_1 & X_2 & \cdots & X_n \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \underline{\hspace{4cm}}$$

5.4 Special Types of Matrices

Certain special types of matrices arise regularly in regression analysis. We consider the most important of these.

Symmetric Matrix

1. If ___________, $\mathbf{A}$ is said to be symmetric.

2. A symmetric matrix necessarily is __________.

3. Symmetric matrices arise typically in regression analysis when we premultiply a matrix, say, **X**, by its transpose, **X**'. The resulting matrix, ________, is symmetric.

Diagonal Matrix

1. A diagonal matrix is a square matrix whose ______________ elements are all _________.

2. We will often not show all zeros for a diagonal matrix, presenting it in the form:

$$\mathbf{B} = \begin{bmatrix} 4 & & & \\ & 1 & & \\ & & 10 & \\ & & & 5 \end{bmatrix}$$

3. **Identity Matrix** The identity matrix or ________ matrix is denoted by ______. It is a diagonal matrix whose elements on the main diagonal are all 1s.

4. Premultiplying or postmultlying any $r \times r$ matrix **A** by the $r \times r$ identity matrix **I** leaves **A** unchanged.

$$\mathbf{AI} = \underline{\hspace{3cm}}$$

5. A **scalar matrix** is a diagonal matrix whose ________________ elements are the _________. A scalar matrix can be expressed as ______, where k is the scalar.

6. Multiplying an $r \times r$ matrix **A** by the $r \times r$ scalar matrix $k\mathbf{I}$ is equivalent to multiplying **A** by the scalar k.

Vector and Matrix with All Elements Unity

1. A column vector with all elements 1 will be denoted by ______ and a square matrix with all elements 1 will be denoted by ______.

2. Note that for an $n \times 1$ vector $\mathbf{1}$ we obtain:

$$\mathbf{1}'\mathbf{1} = [1\ 1\ \cdots\ 1] \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \underline{\hspace{1.5cm}}$$

   and

$$\mathbf{1}\mathbf{1}' = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} [1\ 1\ \cdots\ 1] = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix} = \underline{\hspace{2cm}}$$

Zero Vector

1. A zero vector is a vector containing only zeros. The zero column vector will be denoted by ______.

5.5 Linear Dependence and Rank of Matrix

Linear Dependence

1. Consider the following matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 5 & 1 \\ 2 & 2 & 10 & 6 \\ 3 & 415 & 1 & \end{bmatrix}$$

   We view $\mathbf{A}$ as being made up of four column vectors. Note that the third column vector is a multiple of the first column vector.

$$\begin{bmatrix} 5 \\ 10 \\ 15 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

   We say that the columns of $\mathbf{A}$ are ______________________. They contain ______________ information, since one column can be obtained as a linear combination of the others.

2. We define the set of c column vectors $\mathbf{C}_1, \cdots, \mathbf{C}_c$ in an $r \times c$ matrix to be linearly dependent if one vector can be expressed as a ____________________ of the others. If no vector in the set can be so expressed, we define the set of vectors to be ____________________.

3. When c scalars $k_1, \cdots, k_c$, not all zero, can be found such that:

$$k_l\mathbf{C}_1 + k_2\mathbf{C}_2 + \cdots + k_c\mathbf{C}_c = \mathbf{0}$$

   where $\mathbf{0}$ denotes the zero column vector, the c column vectors are ____________________. If the only set of scalars for which the equality holds is $k_1 = 0, \cdots, k_c = 0$, the set of c column vectors is ____________________.

4. For our example, $k_1 = 5, k_2 = 0, k_3 = -1, k_4 = 0$ leads to:

$$5\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} - 1\begin{bmatrix} 5 \\ 10 \\ 15 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

   Hence, the column vectors are linearly dependent. Note that some of the k_j equal zero here. For linear dependence, it is only required that not all k_j be zero.

Rank of Matrix

1. The rank of a matrix is defined to be the ____________________ of linearly independent ____________ in the matrix.

2. The rank of a matrix is __________ and can equivalently be defined as the maximum number of linearly independent rows.

3. It follows that the rank of an $r \times c$ matrix cannot exceed ____________, the minimum of the two values r and c.

4. When a matrix is the product of two matrices, its rank cannot exceed the smaller of the two ranks for the matrices being multiplied. Thus, if $\mathbf{C} = \mathbf{AB}$, the rank of $\mathbf{C}$ cannot exceed ________________________.

5.6 Inverse of a Matrix

1. In matrix algebra, the inverse of a matrix **A** is another matrix, denoted by ________, such that

   ____

   where **I** is the identity matrix.

Finding the Inverse

1. An inverse of a square $r \times r$ matrix exists if the ________ of the matrix is _____. Such a matrix is said to be nonsingular or of full rank.

2. An $r \times r$ matrix with rank less than r is said to be ___________ or _________________, and does not have an inverse. The inverse of an $r \times r$ matrix of full rank also has rank r.

3. Finding the inverse of a matrix can often require a large amount of computing. We shall take the approach that the inverse of a 2×2 matrix and a 3×3 matrix can be calculated by hand. For any larger matrix, one ordinarily uses a computer to find the inverse.

4. If

   $$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

   then

   $$\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \underline{\hspace{6cm}}$$

   where _______________, D is called the _______________ of the matrix **A**.

5. If **A** were singular, its determinant would equal ________ and no inverse of **A** would exist.

Regression Example

1. The principal inverse matrix encountered in regression analysis is the inverse of the matrix $\mathbf{X}'\mathbf{X}$.

Question .... (p191)

Find the inverse of the matrix $\mathbf{X}'\mathbf{X}$:

$$\mathbf{X}'\mathbf{X} = \begin{bmatrix} n & \sum X_i \\ \sum X_i & \sum X_i^2 \end{bmatrix}$$

sol:

Uses of Inverse Matrix

1. In matrix algebra, if we have an equation:

$$\mathbf{AY} = \mathbf{C}.$$

We correspondingly premultiply both sides by $\mathbf{A}^{-1}$, assuming $\mathbf{A}$ has an inverse

$$\underline{\hspace{3cm}} = \underline{\hspace{2.5cm}}$$

we obtain the solution:

$$\mathbf{Y} = \underline{\hspace{2.5cm}}.$$

5.7 Some Basic Results for Matrices

We list here, without proof, some basic results for matrices which we will utilize in later work.

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

$$(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$$

$$(\mathbf{AB})\mathbf{C}=\mathbf{A}(\mathbf{BC})$$

$$\mathbf{C}(\mathbf{A}+\mathbf{B})=\mathbf{CA}+\mathbf{CB}$$

$$k(\mathbf{A}+\mathbf{B})=k\mathbf{A}+k\mathbf{B}$$

$$(\mathbf{A}')'=\mathbf{A}$$

$$(\mathbf{A}+\mathbf{B})'=\mathbf{A}'+\mathbf{B}'$$

$$(\mathbf{AB})'=\underline{\hspace{2cm}}$$

$$(\mathbf{ABC})'=\underline{\hspace{2.5cm}}$$

$$(\mathbf{AB})^{-1}=\underline{\hspace{2.5cm}}$$

$$(\mathbf{ABC})^{-1}=\underline{\hspace{3cm}}$$

$$(\mathbf{A}^{-1})^{-1}=\mathbf{A}$$

$$(\mathbf{A}')^{-1}=\underline{\hspace{2.5cm}}$$

5.8 Random Vectors and Matrices

Expectation of Random Vector or Matrix

1. A random vector or a random matrix contains elements that are ____________________. Thus, the observations vector $\mathbf{Y}$ in (5.4) is a random vector since the Y_i elements are random variables.

2. The expected value of $\mathbf{Y}$ is a vector, denoted by $E(\mathbf{Y})$, that is defined as follows:

$$E(\mathbf{Y})=\underline{\hspace{2.5cm}},\ i=1,\cdots,n.$$

3. For the error terms in regression model, we have

$$\underline{\hspace{3cm}}.$$

Variance-Covariance Matrix of Random Vector

1. The variance-covariance matrix of $\mathbf{Y}$, denoted by $\sigma^2(\mathbf{Y})$:

$$\sigma^2(\mathbf{Y}) = \underline{\hspace{5cm}}$$

$$= \begin{bmatrix} \sigma^2(Y_1) & \sigma^2(Y_1, Y_2) & \cdots & \sigma^2(Y_1, Y_n) \\ \\ \sigma^2(Y_2, Y_1) & \sigma^2(Y_2) & \cdots & \sigma^2(Y_2, Y_n) \\ \vdots & \vdots & & \vdots \\ \sigma^2(Y_n, Y_1) & \sigma^2(Y_n, Y_2) & \cdots & \sigma^2(Y_n, Y_n) \end{bmatrix}$$

2. Note that the ____________________ are on the main diagonal, and the ________________________ is found in the ith row and jth column of the matrix.

3. The error terms in regression model have constant variance:

$$\sigma^2(\boldsymbol{\varepsilon}) = \underline{\hspace{1.5cm}}.$$

Some Basic Results

1. Frequently, we shall encounter a random vector $\mathbf{W}$ that is obtained by premultiplying the random vector $\mathbf{Y}$ by a constant matrix $\mathbf{A}$ (a matrix whose elements are fixed): $\mathbf{W} = \mathbf{AY}$. Some basic results for this case are:

$$E(\mathbf{A}) = \underline{\hspace{1.5cm}}$$

$$E(\mathbf{W}) = E(\mathbf{AY}) = \underline{\hspace{2.5cm}}$$

$$\sigma^2(\mathbf{W}) = \sigma^2(\mathbf{AY}) = \underline{\hspace{3cm}},$$

where $\sigma^2(\mathbf{Y})$ is the variance-covariance matrix of $\mathbf{Y}$.

✐ Question .... (p42)

Suppose that a random vector $\mathbf{W}$ that is obtained by premultiplying the random vector $\mathbf{Y}$ by a constant matrix $\mathbf{A}$, that is $\mathbf{W} = \mathbf{AY}$. Find the expected value and the variance-covariance matrix of $\mathbf{W}$.

sol:

Multivariate Normal Distribution

1. The density function of the multivariate normal distribution can now be stated as follows:

$$f(\mathbf{Y}) = \underline{\hspace{7cm}},$$

where $\mathbf{Y}$ containing an observation on each of the p Y variables

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_p \end{bmatrix}.$$

2. The mean vector $E(\mathbf{Y})$, denoted by ______, contains the expected values for each of the p Y variables:

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix}.$$

3. The variance-covariance matrix $\sigma^2(\mathbf{Y})$ is denoted by ______: and contains as always the variances and covariances of the p Y variables:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2p} \\ \vdots & \vdots & & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_p^2 \end{bmatrix}$$

σ_i^2 denotes the variance of Y_1, σ_{ij} denotes the covariance of Y_i and Y_j.

4. The multivariate normal density function has properties that correspond to the ones described for the ____________ normal distribution.

5. For instance, if $Y_1, \cdots, Y_p$ are jointly normally distributed (i.e., they follow the multivariate normal distribution), the marginal probability distribution of each variable Y_k is normal, with mean μ_k and standard deviation σ_k.

5.9 Simple Linear Regression Model in Matrix Terms

1. The normal error regression model (2.1):

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i, \quad i = 1, \cdots, n$$

2. The normal error regression model in matrix terms:

__________________________,

where

$\mathbf{Y} =$ ____________ , $\mathbf{X} =$ ______________ , $\boldsymbol{\beta} =$ ____________ , $\boldsymbol{\varepsilon} =$ ____________ ,

$\boldsymbol{\varepsilon}$ is a vector of independent normal random variables with $E(\boldsymbol{\varepsilon}) = \mathbf{0}$ and $\sigma^2(\boldsymbol{\varepsilon}) = \sigma^2\mathbf{I}$

5.10 Least Squares Estimation of Regression Parameters

Normal Equations

Question .... (p200)

Express the normal equations (1.9),

$$\begin{aligned} nb_0 + b_1 \sum X_i &= \sum Y_i \\ b_0 \sum X_i + b_1 \sum X_i^2 &= \sum X_i Y_i \end{aligned}$$

in the matrix form

$$\mathbf{X}'\mathbf{X}\mathbf{b} = \mathbf{X}'\mathbf{Y}$$

where **b** is the vector of the least squares regression coefficients:

$$\mathbf{b}_{2\times 1} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

sol:

✐ Question .... (p201)

Derive the normal equations by the method of least squares in matrix notation.

sol:

Estimated Regression Coefficients

1. Obtain the estimated regression coefficients from the normal equations (5.59) by matrix methods, We premultiply both sides by

   ____

   We then find, since $(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X} = \mathbf{I}$ and $\mathbf{Ib} = \mathbf{b}$,

   $$\mathbf{b} = \underline{\hspace{4cm}}$$

✎ Question .... (p200)

Use matrix methods to obtain the estimated regression coefficients for the Toluca Company example.

sol:

5.11 Fitted Values and Residuals

Fitted Values

1. Let the vector of the fitted values Y_i be denoted by $\hat{\mathbf{Y}}$, then

$$\hat{\mathbf{Y}} = \underline{\hspace{2cm}}$$

$$\begin{bmatrix} \hat{Y}_1 \\ \hat{Y}_2 \\ \vdots \\ \hat{Y}_n \end{bmatrix} = \begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_n \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} b_0 + b_1 X_1 \\ b_0 + b_1 X_2 \\ \vdots \\ b_0 + b_1 X_n \end{bmatrix}$$

2. **Hat Matrix** We can express the matrix result for $\hat{\mathbf{Y}}$ as follows by using the expression for $\mathbf{b}$ in (5.60):

$$\hat{\mathbf{Y}} = \underline{\hspace{4cm}}$$

or, equivalently:

$$\hat{\mathbf{Y}} = \underline{\qquad\qquad}$$

where

$$\mathbf{H}_{n\times n} = \underline{\qquad\qquad\qquad\qquad}$$

3. The fitted values $\hat{Y}_i$ can be expressed as linear combinations of the response variable observations Y_i, with the coefficients being elements of the matrix $\mathbf{H}$.

4. The $\mathbf{H}$ matrix involves only the observations on the predictor variable $\mathbf{X}$. The square $n \times n$ matrix $\mathbf{H}$ is called the **Hat matrix**. It plays an important role in diagnostics for regression analysis (Chapter 10) when we consider whether regression results are unduly influenced by one or a few observations.

5. The matrix $\mathbf{H}$ is symmetric and has the special property (called ________________):

   $$\underline{\qquad\qquad\qquad}$$

   In general, a matrix $\mathbf{M}$ is said to be ______________ if $\mathbf{MM} = \mathbf{M}$.

Residuals

1. Let the vector of the residuals $e_i = Y_i - \hat{Y}_i$ be denoted by $\mathbf{e}$:

   $$\mathbf{e}_{n\times 1} = \underline{\qquad\qquad\qquad\qquad\qquad}$$

2. **Variance-Covariance Matrix of Residuals**. The residuals e_i, like the fitted values $\hat{Y}_i$, can be expressed as linear combinations of the response variable observations Y_i , using the result in (5.73) for $\hat{\mathbf{Y}}$:

   $$\mathbf{e} = \underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

   We thus have the important result:

   $$\mathbf{e} = \underline{\qquad\qquad\qquad}$$

   where $\mathbf{H}$ is the hat matrix defined in (5.53a). The matrix $\mathbf{I} - \mathbf{H}$, like the matrix $\mathbf{H}$, is symmetric and idempotent.

3. The variance-covariance matrix of the vector of residuals $\mathbf{e}$ involves the matrix $\mathbf{I}-\mathbf{H}$:

$$\sigma^2(\mathbf{e}) = \underline{\hspace{4cm}}$$

and is estimated by:

$$s^2(\mathbf{e}) = \underline{\hspace{5cm}}$$

Question .... (p204)

Show that the variance-covariance matrix of $\mathbf{e}$ is $\sigma^2(\mathbf{e}) = \sigma^2(\mathbf{I} - \mathbf{H})$.

sol:

5.12 Analysis of Variance Results

Sums of Squares

Question .... (p42)

Express the sums of squares, $SSTO$, SSE and SSR in matrix notation.

sol:

Sums of Squares as Quadratic Forms

1. In general, a quadratic form is defined as:

$$\underline{\hspace{6cm}}, \quad \text{where} \quad a_{ij} = a_{ji}.$$

2. **A** is a symmetric $n \times n$ matrix and is called the matrix of the quadratic form.

3. The ANOVA sums of squares $SSTO$, SSE, and SSR are all __________________, as can be seen by reexpressing $\mathbf{b}'\mathbf{X}'$.

✎ Question .... (p42)

Show that the ANOVA sums of squares $SSTO$, SSE, and SSR are all quadratic forms.

sol:

5.13 Inferences in Regression Analysis

Regression Coefficients

Question .... (p42)

(a) Derive the variance-covariance matrix of the simple linear regression coefficients, **b** by matrix methods. (b) Obtain the estimated variance-covariance matrix of **b**.

sol:

Mean Response*

Prediction of New Observation*

TA Class

- **Problems**: 5.5, 5.16, 5.22, 5.24, 5.26
- **Exercises**: 5.31