Regression Analysis (I)

Kutner's Applied Linear Statistical Models (5/E)

Chapter 1: Linear Regression with One Predictor Variable

Thursday 09:10-12:00, 資訊 140301

Han-Ming Wu

Department of Statistics, National Chengchi University

http://www.hmwu.idv.tw

Overview

1. Regression analysis (迴歸分析) is a __________________ that utilizes the relation between two or more __________________ so that a _________ or _________ variable can be predicted from the other, or others.

2. Examples: general form of a regression model __________________:
   (a) Y: the sales of a product, X: the amount of advertising expenditures (支出).
   (b) Y: the performance of an employee on a job, X: a battery of aptitude tests (能力傾向成套測驗, 性向測驗).
   (c) Y: the size of the vocabulary of a child, X_1: age of the child, X_2: amount of education of the parents.
   (d) Y: the length of hospital stay of a surgical patient, X_1: the time in the hospital, X_2: the severity of the operation.

3. In this chapter, we consider the basic ideas of regression analysis and discuss the ______________________ of regression models containing a single predictor variable.

1.1 Relations between Variables

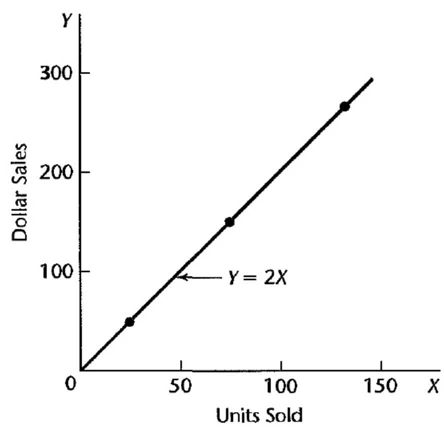
Functional Relation between Two Variables

1. A ____________ relation between two variables is expressed by a ______________ formula. If X denotes the ______________ variable and Y the ____________ variable, a functional relation is of the form:

$$\rule{3cm}{0.4pt}$$

2. **Example**: Y: dollar sales of a product sold at a fixed price, X: the number of units sold. If the selling price is \$2 per unit, the relation is expressed by the equation: __________.

FIGURE 1.1 Example of Functional Relation.



Statistical Relation between Two Variables

1. In general, the observations for a statistical relation do not ______________ the curve of relationship.

2. **Example 1**: Performance evaluations

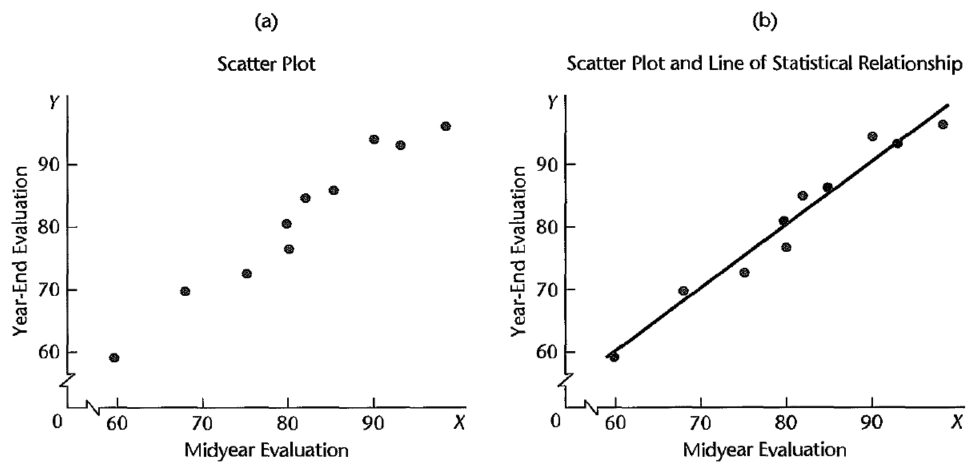
   (a) Performance evaluations for 10 employees were obtained at midyear (X) and at year-end (Y).

   (b) Figure l.2a: the _________ the midyear evaluation, the _________ tends to be the year-end evaluation.

   (c) Figure 1.2b: a __________________ that describes the statistical relation between midyear and year-end evaluations.

(d) Note: that most of the points do not fall directly on the line of statistical relationship. This ____________________ around the line represents ____________ in year-end evaluations that is not associated with midyear performance evaluation and that is usually considered to be of a ________________.

FIGURE 1.2 Statistical Relation between Midyear Performance Evaluation and Year-End Evaluation.



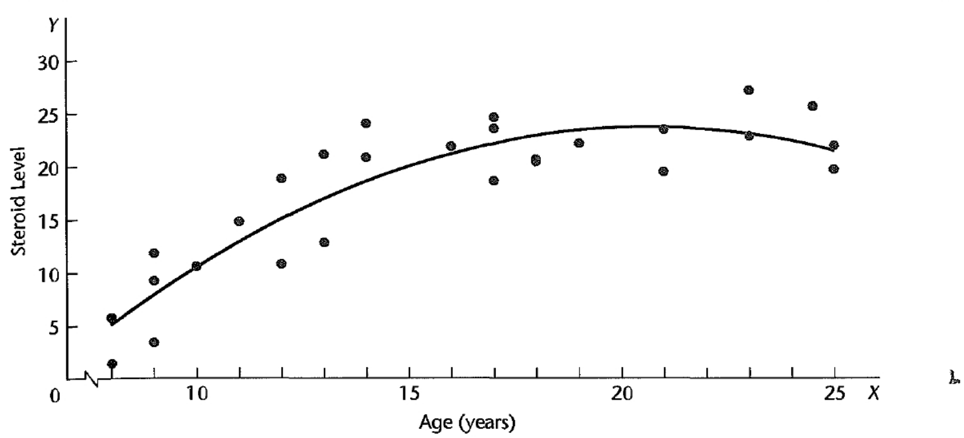
3. **Example 2**:

(a) The data on age and level of a steroid (類固醇) in plasma (血漿) for 27 healthy females between 8 and 25 years old. (Figure 1.3)

(b) The data strongly suggest that the statistical relationship is _____________ (not linear).

(c) As age ____________, steroid level ____________ up to a point and then begins to ____________.

FIGURE 1.3 Curvilinear Statistical Relation between Age and Steroid Level in Healthy Females Aged 8 to 25.



1.2 Regression Models and Their Uses

Historical Origins

1. Regression analysis was first developed by ____________________ in the latter part of the ________________.
2. Galton had studied the relation between ________________________________ and noted that the heights of children of both tall and short parents appeared to ____________ (回復) or ____________ (回歸) to the ____________________.
3. He considered this tendency to be a regression to ________________
4. Galton developed a mathematical description of this _____________________, the precursor of today's regression models.
5. The term regression persists to this day to describe ____________________________________.

> ☺ 行銷資料科學: 小時了了，大未必佳　迴歸均值的有趣現象: `https://medium.com/marketingdatascience/d5f8e5e73163`.
>
> ☺ 均值迴歸 (regression toward the mean) 現象: 當一個特性的極端傾向發生時，會有返回這項特性的平均值 (regression toward mediocrity)。
>
> ☺ 例子: 身高較高的父母，其子女的平均身高，要低於他們父母的平均身高，不會長得更高；相對的，身高比較矮的父母，其子女的平均身高，要高於他們父母的平均身高，不會變得更矮。

Basic Concepts

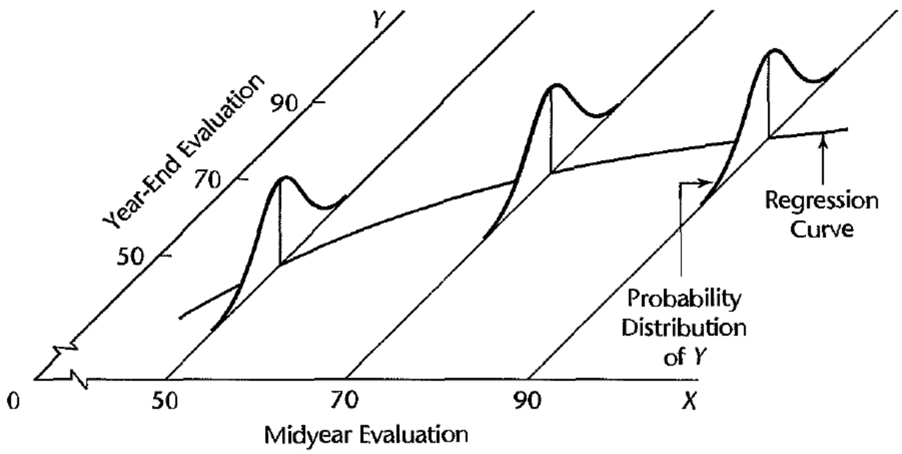
1. A regression model is:
   (a) A tendency of the ____________ variable Y to vary with the ____________ variable X in a ______________ fashion.
   (b) A scattering of points around the _________ of statistical relationship.
2. Assumptions for a regression model:
   (a) There is a _________________________ (機率分佈) of Y for each level of X.

(b) The __________ of these probability distributions vary in some systematic fashion with ______. (___________)

3. **Example**: Performance evaluation (Figure 1.2)

(a) The year-end evaluation Y is treated in a regression model as a __________________. For each level of midyear performance evaluation ______, there is postulated a _____________________________.

FIGURE 1.4 Pictorial Representation of Regression Model.



(b) Figure 1.4: shows probability distributions of Y for midyear evaluation levels at $X = 50$, $X = 70$ and $X = 90$. Note that the __________ of the probability distributions have a systematic relation to the level of X.

(c) This systematic relationship is called the _____________________________. The graph of the regression function is called the __________________.

(d) The regression curve, which describes the relation between __________________ _____________________________ and ________________, is the counterpart to the general tendency of Y to vary with X systematically in a statistical relation.

Construction of Regression Models

1. **Selection of Predictor Variables**:

(a) Choosing a _________________ of explanatory or ____________ variables that is "good" in some sense for the purposes of the analysis.

(b) Other considerations: the ____________ of the variable; the degree to which observations on the variable can be obtained more ____________________ than on competing variables; and the degree to which the variable can be ____________.

2. **Functional Form of Regression Relation**:

   (a) The functional form of the regression relation is __________________ and must be decided upon ____________ once the data have been collected.

   (b) The ________ or __________ regression functions are often used as satisfactory first approximations to regression functions of unknown nature.

3. **Scope of Model**:

   (a) In formulating a regression model, we usually need to __________________ of the model to some interval or region of values of the predictor variable(s).

   (b) **Example**: a company studying the effect of price on sales volume investigated six price levels, ranging from \$4.95 to \$6.95. Here, the scope of the model is limited to price levels ranging from near \$5 to near \$7. The shape of the regression function substantially outside this range would be in serious doubt because the investigation provided no evidence as to the nature of the statistical relation below \$4.95 or above \$6.95.

Uses of Regression Analysis

1. Regression analysis serves three major purposes: (1) ____________, (2) __________, and (3) ____________.

2. The several purposes of regression analysis frequently __________ in practice.

Regression and Causality (因果關係)

1. The existence of a statistical relation between the response variable Y and the explanatory or predictor variable X ______________ in any way that Y depends __________ on X.

2. No matter how strong is the statistical relation between X and Y, no __________________ pattern is necessarily implied by the regression model.

3. **Example**: data on size of vocabulary (X) and writing speed (Y) for a sample of young children aged 5-10 will show a positive regression relation. This relation does not imply, however, that an increase in vocabulary causes a faster writing speed. Here, other explanatory variables, such as age of the child and amount of education, affect both the vocabulary (X) and the writing speed (Y). Older children have a larger vocabulary and a faster writing speed.

4. Regression analysis by itself provides _________________ about causal patterns and must be supplemented by ____________________ to obtain insights about causal relations.

Use of Computers

1. Regression analysis often entails lengthy and tedious calculations, computers are usually utilized to perform the necessary calculations.

2. Almost every statistics package for computers contains a regression component: BMDP, MINITAB, ________, _________, SYSTAT, JMP, S-Plus, MATLAB, and ______.

1.3 Simple Linear Regression Model with Distribution of Error Terms Unspecified

Formal Statement of Model

1. A simple linear regression model:

   ____________________ (1.1)

   where:

   (a) Y_i: the value of the ___________ variable in the ___________.

   (b) β_0 and β_1: ______________ to be estimated.

(c) X_i: the value of the ____________ variable in the ith trial

(d) ϵ_i: a ________________ term with mean ____________ and variance ______________.

(e) ϵ_i and ϵ_j are _______________ so that their covariance is zero (i.e., _______________ for all $i, j; i \neq j$) $i = 1, \cdots, n$.

2. Regression model (1.1) is said to be

(a) simple: there is ____________ predictor variable

(b) linear in the ______________: no parameter appears as an exponent or is multiplied or divided by another parameter

(c) linear in the ____________ variable: because this variable appears only in the first power.

3. A model that is linear in the parameters and in the predictor variabie is also called _____________ model.

Important Features of Model

1. The response Y_i in the ith trial is the sum of two components: (1) the constant term _____________ and (2) the random term ______. Hence, Y_i is a ___________________.

2. Since $E(\epsilon_i) = 0$, it follows that:

$$E(Y_i) = \underline{\hspace{4cm}} = \underline{\hspace{4cm}} = \underline{\hspace{3cm}}.$$

Thus, the response Y_i, when the level of X in the ith trial is X_i, comes from a probability distribution whose mean is:

$$\underline{\hspace{4cm}}.$$

The regression function for model (1.1) is:

$$\underline{\hspace{4cm}}$$

since the regression function relates the means of the probability distributions of Y for given X to the level of X.

3. The response Y_i in the ith trial ____________________ of the value of the regression function (_________) by the error term amount ______.

4. The error terms ϵ_i are assumed to have constant variance _______. It therefore follows that the responses Y_i have the same constant variance:

$$\sigma^2(Y_i) = \sigma^2$$

Thus, regression model (1.1) assumes that the probability distributions of Y have the same variance _______, regardless of the level of the predictor variable X.

5. Since the error terms ϵ_i and ϵ_j are assumed to be uncorrelated, so are the responses _____________.

6. **Summary**: regression model ____________________ implies that the responses Y_i come from probability distributions whose means are ____________________ and whose variances are _______, the same for all levels of X. Further, any two responses Y_i and Y_j are ________________.

7. **Example**: Electrical distributor (Figure 1.6)
A consultant for an electrical distributor is studying the relationship between the number of bids (_______) requested by construction contractors for basic lighting equipment during a week and the number of hours (_______) required to prepare the bids.

   (a) Suppose that regression model (1.1) is:

   $$Y_i = 9.5 + 2.1X_i + \epsilon_i$$

   (b) The regression function is:

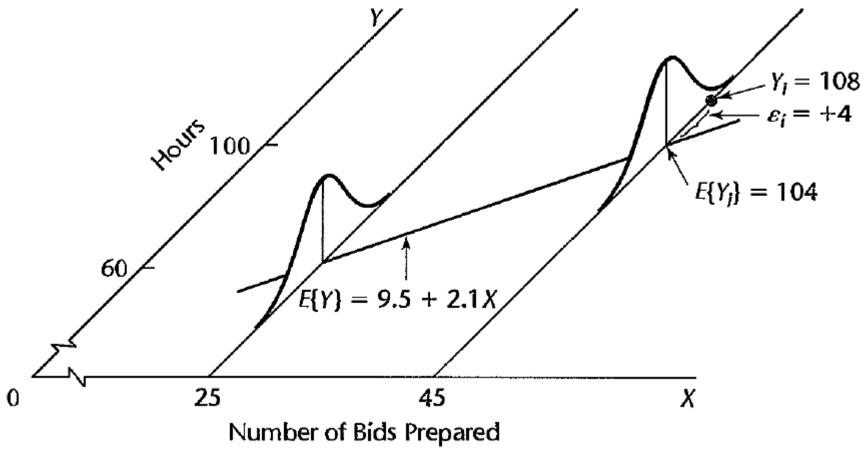
   ____

   (c) Suppose that in the ith week, $X_i = 45$ bids are prepared and the actual number of hours required is $Y_i = 108$. We have

   $E(Y_i) =$ ____________________ and $\epsilon_i =$ __________________________

   (d) The error term ϵ_i is simply the ____________ of Y_i from its mean value $E(Y_i)$.

FIGURE 1.6 **Illustration of Simple Linear Regression Model (1.1).**



Meaning of Regression Parameters

1. The parameters β_0 and β_1, in regression model (1.1) are called ______________________.

   (a) The parameter β_0 is the Y ____________ of the regression line. β_1, is the _________ of the regression line.

   (b) β_1 indicates the __________ in the mean of the probability distribution of Y per unit increase in X.

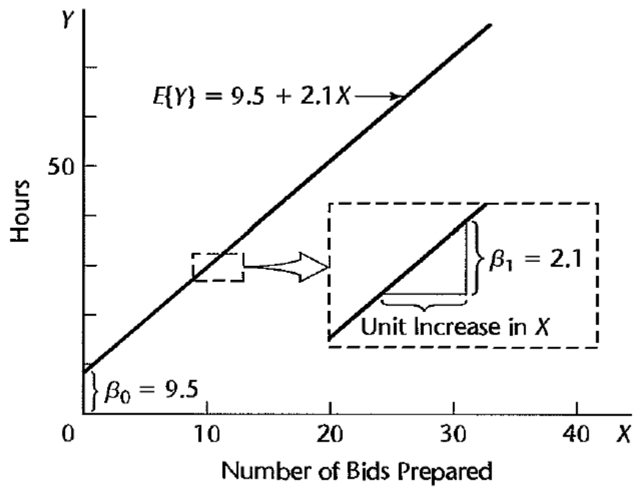
   (c) When the scope of the model includes __________, β_0 gives the mean of the probability distribution of Y at $X = 0$. When the scope of the model does not cover $X = 0$, β_0 ____________________________________ as a separate term in the regression model.

2. **Example**: Electrical distributor (Figure 1.7)

   (a) The regression function: $E(Y) = 9.5 + 2.1X$. The slope $\beta_1, = 2.1$ indicates that the preparation of ________________ bid in a week leads to an ___________ in the _________ of the probability distribution of Y of 2.1 hours.

   (b) The intercept $\beta_0 = 9.5$ indicates the value of the regression function at __________. Since the linear regression model was formulated to apply to weeks where the number of bids prepared ranges from ___________, $\beta_0 = 9.5$ does not have any intrinsic meaning of its own here.

FIGURE 1.7 **Meaning of Parameters of Simple Linear Regression Model (1.1).**



Alternative Versions of Regression Model

1. Let _______ be a constant identically equal to _____. Then, we can write (1.1) as follows:

$$\underline{\hspace{6cm}} \quad \text{where} \quad X_0 \equiv 1$$

   This version of the model associates an X variable with each regression coefficient.

2. An alternative modification is to use for the predictor variable the ____________________ rather than X_i:

$$\begin{aligned} Y_i &= \underline{\hspace{7cm}} \\ &= \underline{\hspace{7.5cm}} \\ &= \underline{\hspace{6cm}}, \end{aligned}$$

   where

$$\beta_0^* = \underline{\hspace{3cm}}$$

1.4 Data for Regression Analysis*

1.5 Overview of Steps in Regression Analysis*

1.6 Estimation of Regression Function

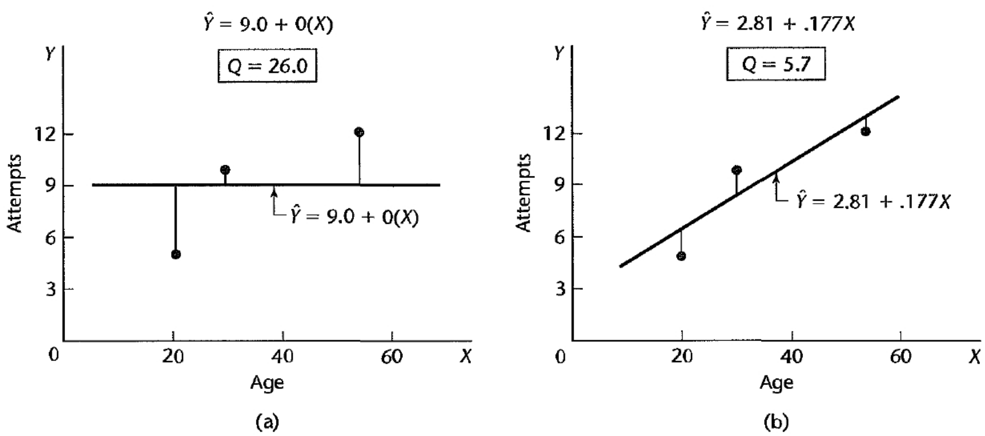
Method of least Squares

1. For the observations __________ for each case, the method of least squares considers the sum of the n squared deviation of Y_i from its expected value $E(Y_i)$:

$$Q = \qquad\qquad\qquad\qquad (1.8)$$

2. According to the method of least squares, the estimators of β_0 and $\beta_!$ are those values b_0 and b_1 respectively, that ____________________ for the given sample observations $(X_1, Y_1), (X_2, Y_2), \cdots, (X_n, Y_n)$.

FIGURE 1.9 **Illustration of Least Squares Criterion Q for Fit of a Regression Line—Persistence Study Example.**



3. **Example**: (Figure 1.9)

   (a) Figure 1.9a: $Y = 9.0 + 0 \cdot X$. This regression line is not a good fit. The sum of the squared deviations for the three cases is:

$$Q = (5 - 9.0)^2 + (12 - 9.0)^2 + (10 - 9.0)^2 = 26.0$$

(b) Figure 1.9b: $Y = 2.81 + 0.177X$ (the least squares regression line). The criterion Q is much reduced:

$$Q = (5 - 6.35)^2 + (12 - 12.55)^2 + (10 - 8.12)^2 = 5.7$$

Thus, a better fit of the regression line to the data corresponds to a smaller sum Q.

4. **Least Squares Estimators**:

(a) For given sample observations (X_i, Y_i), the quantity Q in (1.8) is a function of β_0 and β_1. The values of β_0 and β_1, that minimize Q can tie derived by differentiating (1.8) with respect to β_0 and β_1:

$$\begin{aligned} \frac{\partial Q}{\partial \beta_0} &= \underline{\hspace{6cm}} \\ \frac{\partial Q}{\partial \beta_1} &= \underline{\hspace{6.5cm}} \end{aligned}$$

(b) Set these partial derivatives equal to zero, using b_0 and b_1 (or $\underline{\hspace{3cm}}$) to denote the particular values of β_0 and β_1, that minimize Q:

$$\begin{aligned} -2\sum(Y_i - b_0 - b_1X_i) = 0 \ &\Rightarrow \ \underline{\hspace{6.5cm}} \\ -2\sum X_i(Y_i - b_0 - b_1X_i) = 0 \ &\Rightarrow \ \underline{\hspace{8cm}}. \end{aligned}$$

(c) Normal equations:

$$\underline{\hspace{6cm}}$$

$$\underline{\hspace{7.5cm}},$$

b_0 and b_1 are called point estimators of β_0 and β_1, respectively.

(d) The normal equations can be solved simultaneously for b_0 and b_1:

$$b_0 = \frac{\qquad}{\underline{\hspace{6cm}}} = \underline{\hspace{3cm}}$$

$$b_1 = \frac{\qquad}{\underline{\hspace{5.5cm}}}$$

where $\bar{X}$ and $\bar{Y}$ are the means of the X_i and the Y_i observations, respectively.

5. **Properties of Least Squares Estimators**:

   (a) **Gauss-Markov theorem**: Under the conditions of regression model (1.1), the least squares estimators b_0 and b_1 in (1.10) are ____________ and have ____________________ among all unbiased linear estimators.

   $$\underline{\qquad\qquad\qquad} \quad \text{and} \quad \underline{\qquad\qquad\qquad},$$

   so that neither estimator tends to overestimate or underestimate systematically.

   (b) The theorem states that the estimators b_0 and b_l are ______________ (i.e., their sampling distributions are ______________) than any other estimators belonging to the class of unbiased estimators that are linear functions of the observations $Y_1, \cdots, Y_n$.

   (c) The estimators b_0 and b_1 are such linear functions of the Y_i.

   $$b_1 = \frac{\sum(X_i - \bar{X})(Y_i - \bar{Y})}{\sum(X_i - \bar{X})^2}$$

   This expression is equal to:

   $$b_1 = \frac{\qquad\qquad}{\underline{\qquad\qquad\qquad}} = \underline{\qquad\qquad}$$

   where:

   $$k_i = \frac{\qquad}{\underline{\qquad\qquad\qquad}}$$

   Since the k_i are known constants (because the X_i are known constants), b_1 is a linear combination of the Y_i and hence is a linear estimator.

   (d) In the same fashion, it can be shown that b_0 is a linear estimator.

6. **Example**: The Toluca Company Manufactures Refrigeration Equipment
   In the past, one of the replacement parts has been produced periodically in lots of varying sizes. When a cost improvement program was undertaken, company officials wished to determine the optimum lot size for producing this part. The production of this part involves setting up the production process and machining and assembly operations. One key input for the model to ascertain the optimum lot size was the relationship between lot size and labor hours required to produce the

lot. To determine this relationship, data on lot size and work hours for 25 recent production runs were utilized. The production conditions were stable during the six-month period in which the 25 runs were made and were expected to continue to be the same during the next three years, the planning period for which the cost improvement program was being conducted.

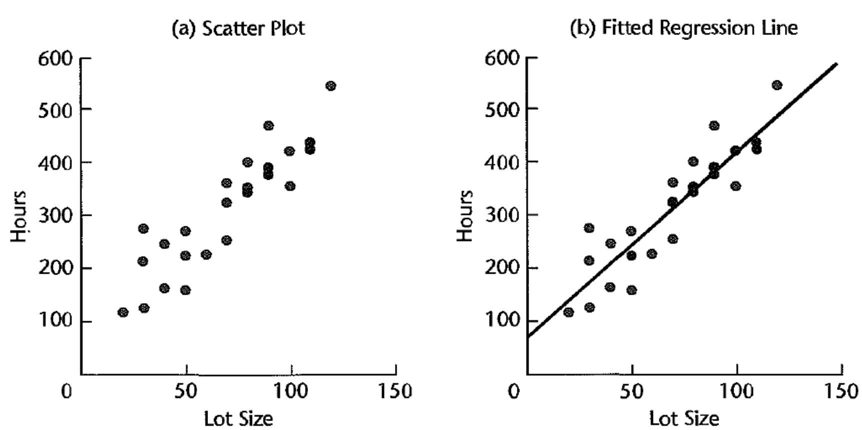
(a) (Table 1.1) All lot sizes are multiples of 10, a result of company policy to facilitate the administration of the parts production.

TABLE 1.1 Data on Lot Size and Work Hours and Needed Calculations for Least Squares Estimates—Toluca Company Example.

Run i	(1) Lot Size X_i	(2) Work Hours Y_i	(3) $X_i - \bar{X}$	(4) $Y_i - \bar{Y}$	(5) $(X_i - \bar{X})(Y_i - \bar{Y})$	(6) $(X_i - \bar{X})^2$	(7) $(Y_i - \bar{Y})^2$
1	80	399	10	86.72	867.2	100	7,520.4
2	30	121	−40	−191.28	7,651.2	1,600	36,588.0
3	50	221	−20	−91.28	1,825.6	400	8,332.0
···	···	···	···	···	···	···	···
23	40	244	−30	−68.28	2,048.4	900	4,662.2
24	80	342	10	29.72	297.2	100	883.3
25	70	323	0	10.72	0.0	0	114.9
Total	1,750	7,807	0	0	70,690	19,800	307,203
Mean	70.0	312.28					

(b) (Figure 1.10a) shows a SYSTAT scatter plot of the data. The scatter plot indicates that the relationship between __________ and ______________ is reasonably _________. We also see that no observations on work hours are ________________________, with reference to the relationship between lot size and work hours.

FIGURE 1.10 SYSTAT Scatter Plot and Fitted Regression Line—Toluca Company Example.



(c) Calculate the least squares estimates:

$$\begin{aligned} b_1 &= \frac{\sum(X_i - \bar{X})(Y_i - \bar{Y})}{\sum(X_i - \bar{X})^2} = \frac{70690}{19800} = 3.5702 \\ b_0 &= \bar{Y} - b_1\bar{X} = 312.28 - 3.5702(70.0) = 62.37 \end{aligned}$$

(d) We estimate that the __________ number of work hours ___________________________ for each additional unit produced in the lot. This estimate applies to the range of lot sizes (from about _______ to about ________) in the data from which the estimates were derived.

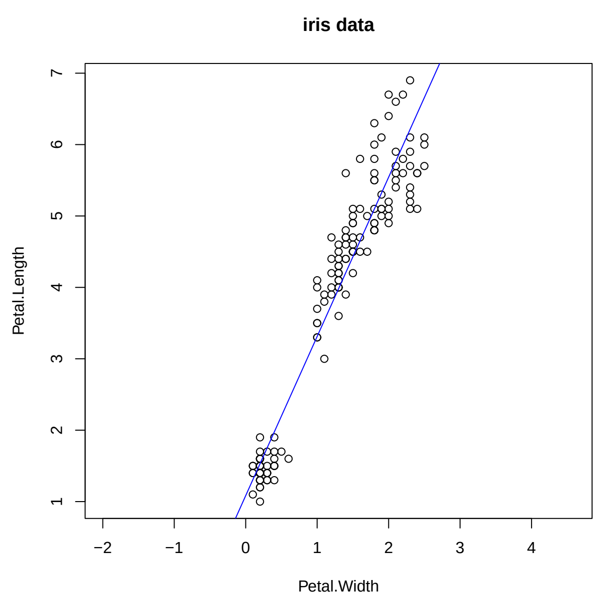
FIGURE 1.11 Portion of MINITAB Regression Output—Toluca Company Example.

```
The regression equation is
Y = 62.4 + 3.57 X

Predictor        Coef       Stdev     t-ratio        p
Constant        62.37       26.18        2.38    0.026
X              3.5702      0.3470       10.29    0.000

s = 48.82        R-sq = 82.2%      R-sq(adj) = 81.4%
```

☺ *R code example*:



```
> head(iris)
  Sepal.Length Sepal.Width Petal.Length Petal.Width Species
1          5.1         3.5          1.4         0.2  setosa
2          4.9         3.0          1.4         0.2  setosa
3          4.7         3.2          1.3         0.2  setosa
4          4.6         3.1          1.5         0.2  setosa
5          5.0         3.6          1.4         0.2  setosa
6          5.4         3.9          1.7         0.4  setosa
> str(iris)
'data.frame':   150 obs. of  5 variables:
 $ Sepal.Length: num  5.1 4.9 4.7 4.6 5 5.4 4.6 5 4.4 4.9 ...
 $ Sepal.Width : num  3.5 3 3.2 3.1 3.6 3.9 3.4 3.4 2.9 3.1 ...
 $ Petal.Length: num  1.4 1.4 1.3 1.5 1.4 1.7 1.4 1.5 1.4 1.5 ...
 $ Petal.Width : num  0.2 0.2 0.2 0.2 0.2 0.4 0.3 0.2 0.2 0.1 ...
 $ Species     : Factor w/ 3 levels "setosa","versicolor",..: 1 1 1 1 1 1 1 1 1 1 ...
> attach(iris)
> plot(Petal.Width, Petal.Length, main = "iris data", asp = 1)
> iris.lm <- lm(Petal.Length ~ Petal.Width)
> summary(iris.lm)

Call:
lm(formula = Petal.Length ~ Petal.Width)

Residuals:
     Min       1Q   Median       3Q      Max
-1.33542 -0.30347 -0.02955  0.25776  1.39453

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  1.08356    0.07297   14.85   <2e-16 ***
Petal.Width  2.22994    0.05140   43.39   <2e-16 ***
---
Signif. codes:  0  '***'  0.001  '**'  0.01  '*'  0.05  '.'  0.1  ' '  1

Residual standard error: 0.4782 on 148 degrees of freedom
Multiple R-squared:  0.9271,    Adjusted R-squared:  0.9266
F-statistic:  1882 on 1 and 148 DF,  p-value: < 2.2e-16

> abline(iris.lm, col = "blue")
```

Point Estimation of Mean Response

1. **Estimated Regression Function**

   (a) Given sample estimators b_0 and b_1 of the parameters in the regression function:

   $$\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

   we estimate the regression function as follows:

   $$\_\_\_\_\_\_\_\_\_\_\_\_\_$$

   where $\hat{Y}$ (read __________) is the value of the estimated regression function at the level X of the predictor variable.

   (b) We call a value of the response variable a ____________ and $E(Y)$ the ________________.

   (c) The mean response stands for the mean of the probability distribution of Y corresponding to the level X of the predictor variable.

   (d) $\hat{Y}$ then is a point estimator of the mean response when the level of the predictor variable is X.

   (e) An extension of the Gauss-Markov theorem: $\hat{Y}$ is an ____________ estimator of $E(Y)$, with ____________________ in the class of unbiased linear estimators.

   (f) For the cases in the study, we will call $\hat{Y}_i$:

   $$\hat{Y}_i = \_\_\_\_\_\_\_\_\_\_, \quad i = 1, \ldots, n$$

   the ______________ for the ith case. Thus, the fitted value $\hat{Y}_i$ is to be viewed in distinction to the ____________________.

2. **Example**: The Toluca Company Example

   (a) (Figure 1.10b) The estimated regression function:

   $$\hat{Y} = 62.37 + 3.5702X$$

   It appears to be a good description of the ________________________ between lot size and work hours.

(b) Suppose that we estimate the mean number of work hours (mean response) required when the lot size is $X = 65$ units:

$$\hat{Y} = \underline{\hspace{6cm}} \quad \text{hours}$$

(c) Interpretation: if many lots of 65 units are produced under the conditions of the 25 runs on which the estimated regression function is based, the mean labor time for these lots is about 294 hours.

(d) $\boxed{\text{NOTE}}$ Of course, the labor time for anyone lot of size 65 is likely to fall above or below the mean response because of inherent variability in the production system, as represented by the error term in the model.)

(e) (Table 1.2) The fitted value for the first case $X_1 = 80$ is:

$$\hat{Y}_1 = \underline{\hspace{6cm}} \quad \text{hours}$$

TABLE 1.2 Fitted Values, Residuals, and Squared Residuals—Toluca Company Example.

Run i	(1) Lot Size X_i	(2) Work Hours Y_i	(3) Estimated Mean Response $\hat{Y}_i$	(4) Residual $Y_i - \hat{Y}_i = e_i$	(5) Squared Residual $(Y_i - \hat{Y}_i)^2 = e_i^2$
1	80	399	347.98	51.02	2,603.0
2	30	121	169.47	−48.47	2,349.3
3	50	221	240.88	−19.88	395.2
···	···	···	···	···	···
23	40	244	205.17	38.83	1,507.8
24	80	342	347.98	−5.98	35.8
25	70	323	312.28	10.72	114.9
Total	1,750	7,807	7,807	0	54,825

☺ *R code example*:

```
> predict(iris.lm, list(Petal.Width = c(0.2, 0.4)))
       1        2
1.529546 1.975534
> data.frame(iris.lm$fitted.values, iris.lm$residuals)
    iris.lm.fitted.values iris.lm.residuals
1                1.529546      -0.129546132
2                1.529546      -0.129546132
3                1.529546      -0.229546132
...
8                1.529546      -0.029546132
9                1.529546      -0.129546132
10               1.306552       0.193447918
...
```

3. **Alternative Model**

   (a) When the alternative regression model (1.6) is to be utilized:

   $$\underline{\hspace{5cm}},$$

   the least squares estimator b_1 of β_1 ____________________ as before.

   (b) The least squares estimator of $\beta_0^* = \beta_0 + \beta_1 \bar{X}$ becomes

   $$b_0^* = \underline{\hspace{6cm}}$$

   Hence, the estimated regression function for alternative model (1.6) is:

   $$\underline{\hspace{4cm}}$$

Residuals (殘差)

1. The ith residual is the difference between the ____________________ and the corresponding _________________. This residual is denoted by ______:

   $$e_i = \underline{\hspace{2cm}}$$

2. For regression model (1.1), the residual e_i becomes:

   $$e_i = \underline{\hspace{6cm}}$$

3. (Figure 1.12) The magnitude of a residual is represented by the ____________________ of the Y_i observation from the corresponding point on the estimated regression function (i.e., from the corresponding fitted value $\hat{Y}_i$).

NOTE We need to distinguish between the model error term value __________________ and the residual _______________. The former involves the vertical deviation of Y_i from the unknown true regression line and hence is _____________. On the other hand, the residual is the vertical deviation of Y_i from the fitted value $\hat{Y}_i$ on the estimated regression line, and it is __________.

4. Residuals are highly useful for studying whether a given regression model is _______________ for the data at hand.

Properties of Fitted Regression Line

1. The sum of the residuals is zero:

____

$$\sum e_i = \sum (Y_i - b_0 - b_1 X_i) = \sum Y_i - nb_0 - b_1 \sum X_i$$

NOTE Rounding errors may, of course, be present in any particular case, resulting in a sum of the residuals that does not equal zero exactly.

2. The sum of the squared residuals, _________, is a minimum. This was the requirement to be satisfied in deriving the least squares estimators of the regression parameters.

3. The sum of the observed values Y_i equals the sum of the fitted values $\hat{Y}_i$:

____

4. The sum of the weighted residuals is zero when the residual in the ith trial is weighted by the level of the predictor variable in the ith trial:

____

5. The sum of the weighted residuals is zero when the residual in the ith trial is weighted by the fitted value of the response variable for the ith trial:

$$\underline{\hspace{4cm}}$$

6. The regression line always goes through the point __________.

$$\hat{Y} = \underline{\hspace{9cm}} = \bar{Y}$$

1.7 Estimation of Error Tenns Variance σ^2

Point Estimator of σ^2

1. The variance σ^2 of the ________________ in regression model (1.1) needs to be estimated to obtain an indication of the _____________ of the probability distributions of Y. A variety of _____________ (推論) concerning the regression function and the prediction of Y require an estimate of σ^2.

2. **Single Population**: The estimator of the variance σ^2 is the sample variance s^2:

$$s^2 = \underline{\hspace{5cm}}$$

which is an ____________ estimator of the variance σ^2 of an infinite population. The sample variance is often called a _______________, because a sum of squares has been divided by the appropriate number of ____________________.

3. **Regression Model**

   (a) We need to calculate a ___________________________, but must recognize that the Y_i now come from ____________ probability distributions with ____________ means that depend upon the level X_i. The deviations are the ____________:

$$\underline{\hspace{4cm}}$$

   and the appropriate sum of squares, denoted by ________, is:

$$\text{SSE} = \underline{\hspace{6cm}}$$

   where SSE stands for ______________________ or _________________________.

(b) The sum of squares SSE has _________ degrees of freedom associated with it. Two degrees of freedom are lost because both _____________ had to be estimated in obtaining the estimated means $\hat{Y}_i$. Hence, the appropriate _______________, denoted by MSE or s^2, is:

$$s^2 = \text{MSE} = \frac{\qquad}{\qquad\qquad\qquad\qquad}$$

where MSE stands for ____________________ or ______________________.

(c) It can be shown that MSE is an ____________ estimator of σ^2 for regression model (1.1): ________________.

4. **Example**: The Toluca Company Example

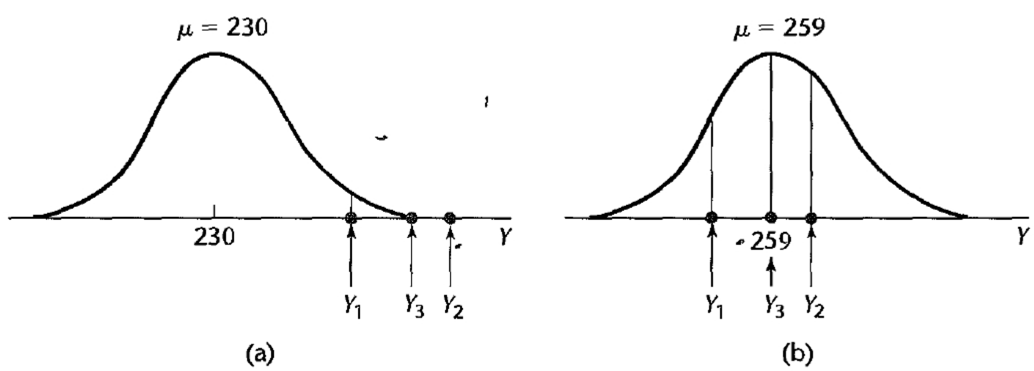
(a) (Table 1.2) we obtain: $SSE = 54,825$ and

$$s^2 = \text{MSE} = \frac{54,825}{23} = 2,384$$

A point estimate of σ, the standard deviation of the probability distribution of Y for any X, is $s = \sqrt{2,384} = 48.8$ hours.

(b) Consider again the case where the lot size is $X = 65$ units. We found earlier that the mean of the probability distribution of Y for this lot size is estimated to be 294.4 hours. Now, we have the additional information that the standard deviation of this distribution is estimated to be 48.8 hours.

FIGURE 1.13 **Densities for Sample Observations for Two Possible Values of μ: $Y_1 = 250$, $Y_2 = 265$, $Y_3 = 259$.**



1.8 Normal Error Regression Model

Model

1. To set up ____________________ and make _________, however, we need to make an assumption about the form of the distribution of the error terms ϵ_i: they are ______________________.

2. The normal error regression model:

$$\underline{\qquad\qquad\qquad\qquad}, \quad i = 1, \cdots, n, \quad (1.24)$$

   (a) Y_i: the ____________________ in the ith trial.

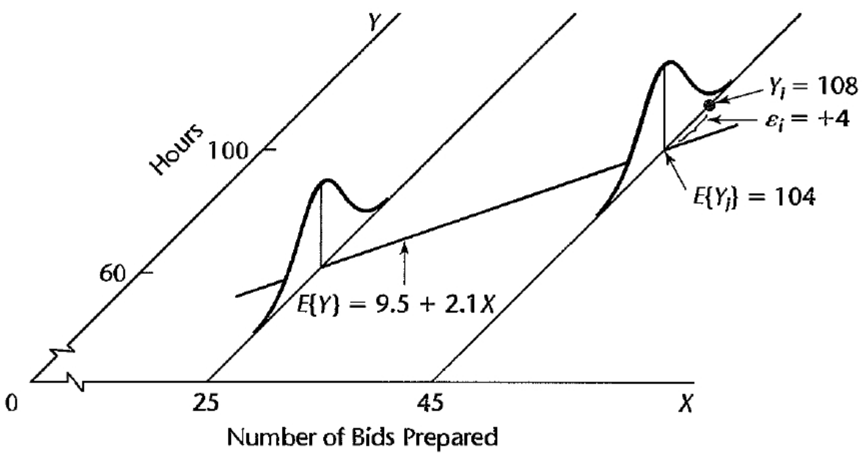
   (b) X_i: known constant, the level of the ____________ variable in the ith trial.

   (c) β_0 and β_1: ______________ to be estimated.

   (d) ϵ_i: independent normally distributed, with mean 0 and variance σ^2 (____________).

3. (Figure 1.6) Regression model (1.24) implies that the ______ are independent normal random variables, with mean ____________________ and variance ______.

FIGURE 1.6 Illustration of Simple Linear Regression Model (1.1).



4. The normality assumption for the error terms is _____________ in many situations because

   (a) the error terms frequently represent the __________________ omitted from the model that ____________________ to some extent and that __________________ without reference to the variable X.

(b) the estimation and testing procedures are based on the ________________ and are usually only sensitive to large departures from _____________. Thus, unless the departures from normality are __________, particularly with respect to ____________, the actual confidence coefficients and risks of errors will be close to the levels for __________________.

Estimation of Parameters by Method of Maximum likelihood

1. **Single Population***

2. **Regression Model**

(a) For the normal error regression model (1.24), each Y_i observation is normally distributed with mean _____________ and standard deviation ______.

(b) The density of an observation Y_i for the normal error regression model (1.24) is:

$$f_i = \underline{\hspace{10cm}}$$

(c) The ____________________ (可能性函數) for n observations $Y_1, Y_2, \cdots, Y_n$ is the product of the individual densities. Since the variance σ^2 of the error terms is usually unknown, the likelihood function is a function of three parameters, ________________.

$$\begin{aligned} L(\beta_0, \beta_1, \sigma^2) &= \underline{\hspace{10cm}} \\ &= \underline{\hspace{10cm}} \end{aligned}$$

(d) The values of β_0, β_1, and σ^2 that maximize this likelihood function are the ______________________________________ (最大概估計量) and are denoted by ________________, respectively.

(e) We find the values of β_0, β_1 and σ^2 that maximize the logarithm of likelihood function $\log L$:

$$\log L = \underline{\hspace{12cm}}.$$

(f) Partial differentiation of the logarithm of the likelihood function:

$$\frac{\partial(\log L)}{\partial\beta_0} = \underline{\hspace{4cm}} = 0$$

$$\frac{\partial(\log L)}{\partial\beta_1} = \underline{\hspace{4.5cm}} = 0$$

$$\frac{\partial(\log L)}{\partial\sigma^2} = \underline{\hspace{5.5cm}} = 0$$

(g) Set these partial derivatives equal to zero, replacing, β_0 , β_1 and σ^2 by the estimators $\hat{\beta}_0$, $\hat{\beta}_1$ and $\hat{\sigma}^2$:

$$\begin{aligned} \underline{\hspace{3.5cm}} &= 0 \\ \underline{\hspace{4cm}} &= 0 \\ \underline{\hspace{3.6cm}} &= \hat{\sigma}^2 \end{aligned}$$

(h) The MLE of β_0 and β_1 are the __________________ as those provided by the method of _______________:

$$\begin{aligned} \hat{\beta}_0 &= \underline{\hspace{2cm}} = b_0 \\ \hat{\beta}_1 &= \underline{\hspace{3.7cm}} = b_1 \\ \hat{\sigma}^2 &= \underline{\hspace{2.5cm}} \end{aligned}$$

(i) The maximum likelihood estimator $\hat{\sigma}^2$ is biased, and ordinarily the unbiased estimator __________________ is used.

$\boxed{\text{NOTE}}$ The unbiased estimator MSE or s^2 differs but slightly from the maximum likelihood estimator $\hat{\sigma}^2$, especially if n is not small:

$$s^2 = MSE = \underline{\hspace{2cm}}.$$

3. **Properties**:
Since the maximum likelihood estimators $\hat{\beta}_0$ and $\hat{\beta}_1$, are the same as the least squares estimators b_0 and b_1 they have the properties of all least squares estimators:

   (a) They are ____________.

   (b) They have ____________________ among all unbiased linear estimators.

   (c) In addition, the maximum likelihood estimators $\hat{\beta}_0$ and $\hat{\beta}_1$, for the normal error regression model have other desirable properties: _____________ (A. 52), ____________ (A.53) and the ____________________________ estimators (linear or otherwise).

😍 TA Class

- **Problems**: 1.6, 1.7, 1.18, 1.20, 1.24
- **Exercises**: 1.32, 1.33, 1.35, 1.36, 1.41
- **Projects**: 1.43