2019/12/23, Calculus Quiz (4), §4.6 ～ §5.6

滿分為 100 分，整體批改標準：說明不清楚都是扣 3 分，符號標錯扣 2 分，其他批改標準於各小題解答後#處。

1.(10%)

3.1 Let $n-1$ points $\{x_1, x_2, \cdots, x_{n-1}\}$ between a and b and satisfying

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

3.2 A **partition** of $[a, b]$: $P = \{x_0, x_1, \cdots, x_{n-1}, x_n\}$.

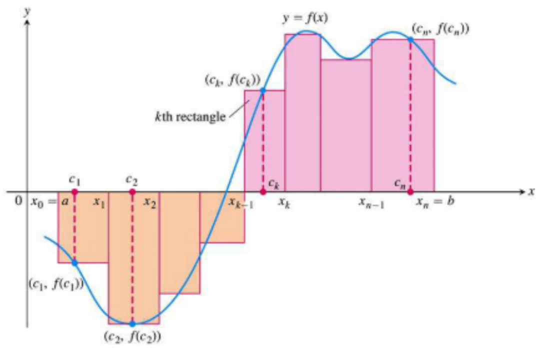
3.3 The kth subinterval of P is $[x_{k-1}, x_k]$.

3.4 The norm of a partition P, $\| P \|$, the largest of all subinterval widths.

3.5 A Riemann sum for f on the interval $[a, b]$:

$$S_P = \sum_{k=1}^{n} f(c_k)\Delta x_k$$

for every $c_k \in [x_{k-1}, x_k]$, $k = 1, \cdots, n$. (圖示如下)



\# 每項佔 2 分

2.(20%)

Consider the partition P that subdivides the interval [a, b] into n subintervals of width $\triangle x = \frac{b-a}{n}$ and let c_k be the right endpoint of each subinterval. So the partition is $P = \{a, a + \frac{b-a}{n}, a + \frac{2(b-a)}{n}, \ldots, a + \frac{n(b-a)}{n}\}$ and $c_k = a + \frac{k(b-a)}{n}$.

We get the Riemann sum $\sum_{k=1}^{n} f(c_k)\triangle x = \sum_{k=1}^{n} c_k^2\left(\frac{b-a}{n}\right) = \frac{b-a}{n}\sum_{k=1}^{n}\left(a + \frac{k(b-a)}{n}\right)^2 = \frac{b-a}{n}\sum_{k=1}^{n}\left(a^2 + \frac{2ak(b-a)}{n} + \frac{k^2(b-a)^2}{n^2}\right)$

$= \frac{b-a}{n}\left(\sum_{k=1}^{n} a^2 + \frac{2a(b-a)}{n}\sum_{k=1}^{n} k + \frac{(b-a)^2}{n^2}\sum_{k=1}^{n} k^2\right) = \frac{b-a}{n} \cdot na^2 + \frac{2a(b-a)^2}{n^2} \cdot \frac{n(n+1)}{2} + \frac{(b-a)^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$

$= (b-a)a^2 + a(b-a)^2 \cdot \frac{n+1}{n} + \frac{(b-a)^3}{6} \cdot \frac{(n+1)(2n+1)}{n^2} = (b-a)a^2 + a(b-a)^2 \cdot \frac{1+\frac{1}{n}}{1} + \frac{(b-a)^3}{6} \cdot \frac{2+\frac{3}{n}+\frac{1}{n^2}}{1}$

As $n \to \infty$ and $\|P\| \to 0$ this expression has value $(b-a)a^2 + a(b-a)^2 \cdot 1 + \frac{(b-a)^3}{6} \cdot 2$

$= ba^2 - a^3 + ab^2 - 2a^2b + a^3 + \frac{1}{3}(b^3 - 3b^2a + 3ba^2 - a^3) = \frac{b^3}{3} - \frac{a^3}{3}$. Thus, $\int_a^b x^2 dx = \frac{b^3}{3} - \frac{a^3}{3}$.

\# 只寫$\frac{b^3}{3} - \frac{a^3}{3}$得 5 分; Riemann sum 佔 10 分; 計算過程與答案佔 10 分

3.(10%)

Theorem 4: The Fundamental Theorem of Calculus Part 1

If f is continuous on $[a, b]$ then

(a) $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b),

(b) and its derivative is $f(x)$;

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Theorem 4: The Fundamental Theorem of Calculus Part 2

If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(x)\big|_a^b = F(b) - F(a).$$

4. (20%)

$$y = \int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}},\ |x| < \frac{\pi}{2} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 x}}\left(\frac{d}{dx}(\sin x)\right) = \frac{1}{\sqrt{\cos^2 x}}(\cos x) = \frac{\cos x}{|\cos x|} = \frac{\cos x}{\cos x} = 1 \text{ since } |x| < \frac{\pi}{2}$$

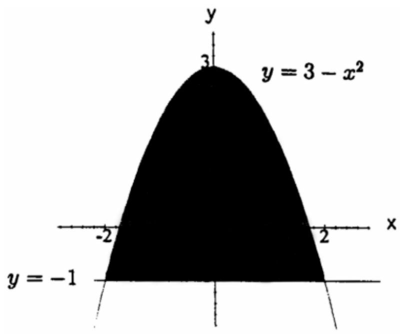
5.(20%)

Let $u = x^3 + 1 \Rightarrow du = 3x^2 dx$ and $x^3 = u - 1$. So $\int 3x^5\sqrt{x^3+1}\,dx = \int (u-1)\sqrt{u}\,du = \int \left(u^{3/2} - u^{1/2}\right)du$
$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C = \frac{2}{5}(x^3+1)^{5/2} - \frac{2}{3}(x^3+1)^{3/2} + C$

變數變換 5%，計算過程+答案 15%

6.(20%)

(a) Limits of integration: $y = 3 - x^2$ and $y = -1$
$\Rightarrow 3 - x^2 = -1 \Rightarrow x^2 = 4 \Rightarrow a = -2$ and $b = 2$;
$f(x) - g(x) = (3 - x^2) - (-1) = 4 - x^2$
$\Rightarrow A = \int_{-2}^{2}(4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$
$= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) = 16 - \frac{16}{3} = \frac{32}{3}$



(b) Limits of integration: let $x = 0$ in $y = 3 - x^2$
$\Rightarrow y = 3$; $f(y) - g(y) = \sqrt{3-y} - \left(-\sqrt{3-y}\right)$
$= 2(3-y)^{1/2}$
$\Rightarrow A = 2\int_{-1}^{3}(3-y)^{1/2}\,dy = -2\int_{-1}^{3}(3-y)^{1/2}(-1)\,dy = (-2)\left[\frac{2(3-y)^{3/2}}{3}\right]_{-1}^{3} = \left(-\frac{4}{3}\right)\left[0 - (3+1)^{3/2}\right]$
$= \left(\frac{4}{3}\right)(8) = \frac{32}{3}$

(a)(b)各佔 10 分，每小題答案對過程錯扣 5 分