THOMAS' CALCULUS (12/E)

10.1 Sequences

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1 Sequences, Convergence and Divergence

1.1 A sequence is __________________(____________________) in a given order.

1.2 Each of $a_1, a_2, \cdots$ are the _________ of the sequence.

1.3 Example: $2, 4, 6, 8, 10, 12, \cdots, 2n, \cdots$ has first term $a_1 = 2$, second term $a_2 = 4$ and ____________ (____________). The integer n is called the _________ of a_n.

1.4 *Definitions: Infinite Sequence*

> An infinite sequence of numbers is a ____________ whose ___________ is the set of __________________.

1.5 The sequence $1, 2, 3, 4, \cdots$ is not the same as the sequence $2, 1, 3, 4, \cdots$; __________ is important.

1.6 Examples:

nth term	listing terms	write
$a_n = \sqrt{n}$	$\{a_n\} =$ ____________________________	$\{a_n\} =$ ____________
$b_n = (-1)^{n+1}\frac{1}{n}$	____	_________________________
$c_n = \frac{n-1}{n}$	________________________________	____________________
$d_n = (-1)^{n+1}$	______________________________________	________________________

1.7 *Definitions: Converges, Diverges, Limit*

The sequence $\{a_n\}$ converges to the number ______ if to every positive number ______ there corresponds an integer _______ such that for all ______,

____

If no such number L exists, we say that $\{a_n\}$ ____________. If $\{a_n\}$ converges to L, we write __________________ or simply ____________ and call L the _________ of the sequence.

1.8 Examples:

(a) $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{n}, \cdots\}$; $\lim_{n\to\infty} a_n =$ ______

(b) $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots, 1-\frac{1}{n}, \cdots\}$; $\lim_{n\to\infty} a_n =$ ______

(c) $\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \cdots, \sqrt{n}, \cdots\}$; ______________

(d) $\{1, -1, 1, -1, \cdots, (-1)^{n+1}, \cdots\}$; ______________

1.9 *Definitions: Diverges to Infinity*

The sequence $\{a_n\}$ ____________ to infinity if for every number _______ there is an integer _______ such that for all ______ larger than N, ____________. If this condition holds we write

___________________ or ____________

Similarly if for every number m there is an integer N such that for all $n > N$ we have $a_n < m$ then we say $\{a_n\}$ diverges to negative infinity and write

_____________________ or ______________

1.10 A sequence may diverge without diverging to infinity or negative infinity. Examples: _____________________ and _______________________________.

2 Calculating Limits of Sequences

2.1 *Theorem 1*

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$

1. Sum Rule: _________________________
2. Difference Rule: _________________________

3. Product Rule: ____________

4. Constant Multiple Rule: ____________

5. Quotient Rule: ____________

2.2 *Theorem 2: The Sandwich Theorem for Sequences*

Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers. If ____________ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n =$ ______ then ____________ also.

2.3 *Theorem 3: The Continuous Function Theorem for Sequences*

Let $\{a_n\}$ be a sequence of real numbers. If ________ and if f is a function that is ____________ and defined at all a_n, then ____________.

✎ Ex. **1** .... (example3, p536)

(a) $\lim_{n\to\infty}(-\frac{1}{n}) =$

(b) $\lim_{n\to\infty}(\frac{n-1}{n}) =$

(c) $\lim_{n\to\infty}\frac{5}{n^2} =$

(d) $\lim_{n\to\infty}\frac{4-7n^6}{n^6+3} =$

✎ Ex. **2** .... (example4, p536)

(a) $\lim_{n\to\infty}(\frac{\cos n}{n}) =$

(b) $\lim_{n\to\infty}(\frac{1}{2^n}) =$

(c) $\lim_{n\to\infty}(-1)^n\frac{1}{n} =$

✐ Ex. **3** .... (example5, p537)

Show that $\sqrt{(n+1)/n} \to 1$.

sol:

✐ Ex. **4** .... (example6, p537)

Find $\lim_{n\to\infty} 2^{1/n}$.

sol:

3 Using L'Hopital's Rule

3.1 *Theorem 4*

Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that is a sequence of real numbers such that __________ for $n \geq n_0$. Then __________ $\Rightarrow$ __________

✐ Ex. **5** .... (example7, p537)

Show that $\lim_{n\to\infty} \frac{\ln n}{n} = 0$.

sol:

Ex. 6 .... (example8, p537)

Does the sequence whose nth term is $a_n = (\frac{n+1}{n-1})^n$ converges? If so, find $\lim_{n\to\infty} a_n$.

sol:

4 Commonly Occurring Limits

4.1 *Theorem 5*

1. $\lim_{n\to\infty} \frac{\ln n}{n} = \underline{\qquad}$
2. $\lim_{n\to\infty} \sqrt[n]{n} = \underline{\qquad}$
3. $\lim_{n\to\infty} x^{1/n} = \underline{\qquad} \quad (x > 0)$
4. $\lim_{n\to\infty} x^n = \underline{\qquad} \quad (|x| < 1)$
5. $\lim_{n\to\infty} (1 + \frac{x}{n})^n = \underline{\qquad} \quad (\text{any}x)$
6. $\lim_{n\to\infty} \frac{x^n}{n!} = \underline{\qquad} \quad (\text{any}x)$

4.2 *Definition*

A sequence $\{a_n\}$ is ______________ if ___________ for all n. That is, _________________. The sequence is ______________ if ___________ for all n. The sequence $\{a_n\}$ is ___________ if it is either nondecreasing or nonincreasing.

4.3 *Theorem 6: The Monotonic Sequence Theorem*

If a sequence $\{a_n\}$ is both __________ and ___________, then the sequence ___________.

✎ Ex. **7** .... (`example9, p538`)

(a) $\lim_{n\to\infty} \frac{\ln(n^2)}{n} =$

(b) $\lim_{n\to\infty} \sqrt[n]{n^2} =$

(c) $\lim_{n\to\infty} \sqrt[n]{3n} =$

(d) $\lim_{n\to\infty} (-\frac{1}{2})^n =$

(e) $\lim_{n\to\infty} (\frac{n-2}{n})^n =$

(f) $\lim_{n\to\infty} \frac{100^n}{n!} =$

實習課練習 (EXERCISE 10.1)

☐ Find the values of a_1, a_2, a_3 and a_4.

3. $a_n = \dfrac{(-1)^{n+1}}{2n-1}$

☐ Write out the first ten terms of the sequence.

9. $a_1 = 2, a_{n+1} = (-1)^{n+1}a_n/2$

☐ Find a formula for the nth term of the sequence.

16. The sequence $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \cdots$

19. The sequence $0, 3, 8, 15, 24, \cdots$

☐ Which of the sequence $\{a_n\}$ converge, and which diverge? Find the limit of each convergent sequence.

27. $a_n = 2 + (0.1)^n$

31. $a_n = \dfrac{1-5n^4}{n^4+8n^3}$

36. $a_n = (-1)^n(1 - \frac{1}{n})$

43. $a_n = \sin(\frac{\pi}{2} + \frac{1}{n})$

54. $a_n = (1 - \frac{1}{n})^n$

57. $a_n = (\frac{3}{n})^{1/n}$

63. $a_n = \dfrac{n!}{n^n}$

71. $a_n = (\frac{x^n}{2n+1})^{1/n}, \quad x > 0$

84. $a_n = \sqrt[n]{n^2+n}$

86. $a_n = \dfrac{(\ln n)^5}{\sqrt{n}}$

89. $a_n = \dfrac{1}{n}\displaystyle\int_1^n \frac{1}{x}\,dx$