THOMAS' CALCULUS (12/E)

8.7 Improper Integrals

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1 Infinite Limits of Integration

1.1 *Definitions: Type I Improper Integrals*

Integrals with __________ of integration are **improper integrals of Type I**.

(a) If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\ dx = \underline{\hspace{4cm}}$$

(b) If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\ dx = \underline{\hspace{4cm}}$$

(c) If $f(x)$ is continuous on $(-\infty, \infty)$ then

$$\int_{-\infty}^\infty f(x)\ dx = \underline{\hspace{6cm}}$$

where c is any real number. In each case, if the limit is ______ we say that the improper integral ________ and that the ______ is the **value** of the improper integral. If the limit fails to exist, the improper integral ________ (the area under the curve is infinite).

1.2 Examples:

(a) Upper limit: $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\ dx = \underline{\hspace{4cm}}$

(b) Lower limit: $\displaystyle\int_{-\infty}^0 \frac{dx}{1+x^2} = \underline{\hspace{4cm}}$

(c) Both limits: $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2} =$ ______________________________ .

Ex. 1 .... (example1, p479)

Is the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ finite? If so, what is it?

sol:

Ex. 2 .... (example2, p479)

Evaluate $\displaystyle\int_{\infty}^{\infty} \frac{dx}{1+x^2}$.

sol:

Ex. 3 .... (example2, p480)

For what values of p does the integral $\displaystyle\int_{1}^{\infty} \frac{1}{x^p}\, dx$ converge? When the integral does converge, what is its value?

sol:

2 Integrands with Vertical Asymptotes

2.1 *Definitions: Type II Improper Integrals*

Integrals of functions that become ________________ within the interval of integration are **improper integrals of Type II**.

(a) If $f(x)$ is continuous on $(a, b]$ and is discontinuous at a then

$$\int_a^b f(x)\, dx = \underline{\hspace{4cm}}$$

(b) If $f(x)$ is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\, dx = \underline{\hspace{4cm}}$$

(c) If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$ then

$$\int_a^b f(x)\, dx = \underline{\hspace{5.5cm}}$$

In each case, if the limit is finite we say the improper integral ____________ and that the limit is the value of the improper integral. If the limit does not exist, the integral ___________.

2.2 Examples:

(a) Upper endpoint: $\displaystyle\int_0^1 \frac{dx}{(x-1)^{2/3}} =$ ________________

(b) Lower endpoint: $\displaystyle\int_1^3 \frac{dx}{(x-1)^{2/3}} =$ ________________

(c) Interior point: $\displaystyle\int_0^3 \frac{dx}{(x-1)^{2/3}} =$ ________________________

Ex. 4 .... (`example4, p482`)

Investigate the convergence of $\displaystyle\int_0^1 \frac{1}{1-x}\, dx$.

sol:

Ex. 5 .... (`example5, p482`)

Evaluate $\displaystyle\int_0^3 \frac{dx}{(x-1)^{2/3}}$.

sol:

3 Tests for Convergence and Divergence

3.1 *Theorem 1: Direct Comparison Test*

Let f and g be continuous on $[a,\infty)$ with ____________________ for all $x \geq a$. Then

(a) $\displaystyle\int_a^\infty f(x)\ dx$ ______________ if $\displaystyle\int_a^\infty g(x)\ dx$ ______________.

(b) $\displaystyle\int_a^\infty g(x)\ dx$ ____________ if $\displaystyle\int_a^\infty f(x)\ dx$ ____________.

3.2 *Theorem 2: Limit Comparison Test*

If the positive functions f and g are continuous on $[a,\infty)$ and if ________________ , $0 < L < \infty$, then ________________ and ________________ both ______________ or both ____________.

Ex. 6 .... (`example6, p483`)

Does the integral $\displaystyle\int_1^\infty e^{-x^2}\ dx$ converges?

sol:

✐ Ex. **7** .... (example7, p484)

(a) $\int_1^\infty \frac{\sin^2 x}{x^2}\, dx$ converges because

(b) $\int_1^\infty \frac{1}{\sqrt{x^2 - 0.1}}\, dx$ diverges because

✐ Ex. **8** .... (example8, p485)

Show that $\int_1^\infty \frac{dx}{1+x^2}$ converges and find the integral value.

sol:

✐ Ex. **9** .... (example9, p485)

Investigate the convergence of $\int_1^\infty \frac{1-e^{-x}}{x}\, dx$.

sol:

實習課練習 (EXERCISE 8.7)

☐ Evaluate the integrals.

2. $\displaystyle\int_1^\infty \frac{dx}{x^{1.001}}$

7. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

10. $\displaystyle\int_{-\infty}^2 \frac{2\ dx}{x^2+4}$

13. $\displaystyle\int_{-\infty}^\infty \frac{2x\ dx}{(x^2+1)^2}$

17. $\displaystyle\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$

22. $\displaystyle\int_0^\infty 2e^{-\theta}\sin\theta\ d\theta$

25. $\displaystyle\int_0^1 x\ln x\ dx$

30. $\displaystyle\int_2^4 \frac{dt}{t\sqrt{t^2-4}}$

32. $\displaystyle\int_0^2 \frac{dx}{\sqrt{|x-1|}}$

☐ Test the integrals for convergence.

39. $\displaystyle\int_0^{\ln x} x^{-2}e^{-1/x}dx$

45. $\displaystyle\int_{-1}^1 \ln|x|\ dx$

54. $\displaystyle\int_2^\infty \frac{x\ dx}{\sqrt{x^4-1}}$

58. $\displaystyle\int_2^\infty \frac{1}{\ln x}\ dx$

61. $\displaystyle\int_1^\infty \frac{1}{\sqrt{e^x-x}}$